Existence of attractor for the quasi-geostrophic approximation of the Navier-Stokes equations and estimate of its dimension

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Existence d'un attracteur pour l'approximation quasi-géostrophique des équations de Navier-Stokes et estimation de sa dimension

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Résumé : On étudie l'approximation quasi-géostrophique des équations de Navier-Stokes. Cette approximation est couramment utilisée pour la modélisation des circulations océaniques. On suppose que l'océan est divisé en N couches. On démontre l'existence et l'unicité de la solution et ensuite l'existence d'un attracteur maximal qui décrit le comportement à long terme des solutions et on déduit des estimations de ses dimensions fractal et de Hausdorff en fonctions des données.

Existence of attractor for the quasi-geostrophic approximation of the Navier-Stokes equations and estimate of its dimension

Abstract : In this paper, we study the quasi-geostrophic approximation of the Navier-Stokes equations. This approximation is usually used in modelisation of oceanic circulations. First, we consider the barotropic modelisation, and in a second part, we supposed that the ocean is divided in N layers. For the both modelisations, we prove the existence and uniqueness of the solution, and then the existence of a maximal attractor which describes the long time behaviour of the solutions and we derive estimates of its Hausdorff and fractal dimensions in terms of the data.

Key-words : Navier-Stokes equations, attractor.

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Introduction :

The important role played by attractors in the study of the large time behaviour of the solutions of partial differential equations is now well known. The existence of a maximal attractor which attracts all trajectories as time goes to infinity has been already proved for the Navier-Stokes equations (Babin-Vishik [1], Constantin-Foias-Temam [6]). In this work, we consider the quasi-geostrophic approximation of these equations, when the unknown functions are stream function and vorticity, with Dirichlet boundary conditions (Pedloski [14], Le Provost [10]).

This paper is divided in four parts. First, we consider the barotropic modelisation, and in parts 3 and 4, the multilayer problem. For both modelisations, we prove -parts 1 and 3- existence and uniqueness of the solution of these equations, using classical arguments (Lions [11]), and then -parts 2 and 4- study the existence of a maximal attractor, and also derive the estimates of the Hausdorff and fractal dimensions (Temam [15], Marion [13]).

Notation :

Let Ω be an open bounded set of $\mathbb{R}^2$ with boundary Γ. For $p \in [1, +\infty]$, we denote by $L^p(\Omega)$ the space of measurable scalar functions on Ω for which

$$|u|_{L^p(\Omega)} = \left(\int |u(x)|^p \right)^{1/p} < +\infty \quad \text{for } 1 \leq p < +\infty .$$

$$|u|_{L^\infty(\Omega)} = \underset{x \in \Omega}{\text{ess sup}} \, |u(x)| < +\infty .$$

We denote by $H^k(\Omega)$ the Sobolev space of scalar functions which are in $L^2(\Omega)$ together with their derivatives of order less than or equal to k. As usual $H^1_0(\Omega)$ is the Hilbert subspace of $H^1(\Omega)$ made of functions vanishing on Γ. We note $|u| = |u|_{L^2(\Omega)}$, (u,v) is the scalar product in $L^2(\Omega)$ and $\langle u,v \rangle$ is the duality pairing in $H^{-1}(\Omega) \times H^1_0(\Omega)$. Let I be a bounded interval of $\mathbb{R}$ and X a Banach space. We denote by $L^p(I,X)$, $1 \leq p \leq +\infty$, the space of measurable functions f from I into X such that $\| f \|_X \in L^p(I)$. This is a Banach space for the norm $\| f \|_{L^p(I,X)} = | \| f \|_X |_{L^p(I)}$.

We denote by λ_1 the first eigenvalue of the Laplacian operator on Ω and for Ω regular, by c_y the constant depending on Ω given by the Youdovitch inequality (Lions [10]) for f in $H^2(\Omega) \cap H^1_0(\Omega)$:

$$\| f \|_{H^2(\Omega)} \leq c_y \, |\Delta f| .$$

TABLE OF CONTENTS.

THE BAROTROPIC MODELISATION.

PART 1. EXISTENCE AND UNIQUENESS OF THE SOLUTION.

1.1. Equations and Theorem.

Let Ω denote a regular enough open bounded set of $\mathbb{R}^2$ with boundary Γ. We consider the following initial boundary value problem involving a scalar function $\xi(x,t)$; ξ satisfies :

$$\frac{\partial \xi}{\partial t} + \varepsilon \xi + \frac{\partial \psi}{\partial x} - A \Delta \xi + J(\psi,\xi) = v \qquad \text{in } \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$\Delta \psi = \xi \qquad \text{in } \Omega \times \mathbb{R}^+, \tag{1.2}$$

$$\xi(.,0) = \xi_0 (.) \qquad \text{in } \Omega, \tag{1.3}$$

$$\psi = 0 \qquad \text{on } \partial\Omega, \tag{1.4}$$

$$\xi = 0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

where $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

We recall that ξ is the vorticity, ψ the stream function. The curl of the wind v is supposed independent of time and given in $L^2(\Omega)$. The coefficients of bottom friction ε and of lateral friction A are positive. The initial data ξ_0 is given in $H^{-1}(\Omega)$.

We equip $H^{-1}(\Omega)$ with the scalar product :

$$<< \xi_1, \xi_2 >> = - < \xi_1, \Delta^{-1} \xi_2 > = \int \nabla\psi_1 . \nabla\psi_2$$

where ψ_i are defined by $\Delta\psi_i = \xi_i$, $\psi_i \in H_0^1(\Omega)$. It defines a norm $| \,.\, |_{-1}$, which is equivalent to the usual norm on $H^{-1}(\Omega)$.

Theorem 1.1. 1) The problem (1.1) - (1.4) admits a unique solution ξ in $\mathscr{C}([0,T], H^{-1}(\Omega)) \cap L^2(0,T,L^2(\Omega)) \cap L^2_{loc}(0,T,H^1_0(\Omega))$ for all $T > 0$,

2) This solution verifies for $t \geq 0$

$$(\alpha) \qquad A \int_0^t |\xi|^2 \leq t\, |v|^2 \inf 1/\lambda_1 + |\nabla\psi(0)|^2$$

$$(\beta) \qquad |\nabla\psi(t)|^2 \leq |\nabla\psi(0)|^2 \exp(-\varepsilon_A t) + \frac{|v|^2 \inf 1}{\lambda_1 \varepsilon_A} (1 - \exp(-\varepsilon_A t))$$

and for $0 < t < T$

$$(\gamma) \qquad t\, |\xi(t)|^2 \leq K_1 |\nabla\psi(0)|^2 + K_2$$

with $\inf 1 = \inf \left(\frac{1}{\varepsilon}, \frac{1}{A\lambda_1}\right)$, $\varepsilon_A = (\varepsilon + A\,\lambda_1)$, K_1 and K_2 real constants depending on Ω, A, ε, ν, T.

3) The mapping $\xi_0(.) \to \xi(t,.)$ is continuous in $H^{-1}(\Omega)$.

1.2. Proof of theorem.

This proof relies on classical arguments (Lions [11]) and we shall only give its main steps. We implement a Galerkin method using the orthogonal basis $\{w_j\}_j$ of $H_0^1(\Omega)$ defined by $-\Delta w_j = \lambda_j w_j$ in Ω. We denote by V_m the space spanned by $\{w_1,...,w_m\}$. For each integer m, we look for an approximate solution of the form :

$$\psi_m(.,t) = \sum_{i=1}^{m} g_{i,m}(t)w_i \text{ and } \xi_m(.,t) = \sum_{i=1}^{m} -\lambda_i g_{i,m}(t)w_i = \sum_{i=1}^{m} h_{i,m}(t)w_i$$

satisfying in $\Omega \times \mathbb{R}^+$:

(1.6) $\left(\frac{\partial \xi_m}{\partial t}, w_j\right) + \varepsilon(\xi_m, w_j) + \left(\frac{\partial \psi_m}{\partial x}, w_j\right) + A\,(\nabla\xi_m, \nabla w_j) + (J(\psi_m, \xi_m), w_j) = (\nu, w_j)$

(1.7) $(\nabla\psi_m, \nabla w_j) = -(\xi_m, w_j)$

(1.8) $\xi_m(.,0) = \xi_{m,0}(.)$ in Ω.

and

(1.9) $\xi_{m,0} \to \xi_0$ in $H^{-1}(\Omega)$ when $m \to +\infty$, $|\xi_{m,0}|_{-1} \le |\xi_0|_{-1}$.

Observe that (1.7) is equivalent to $\Delta\psi_m = \xi_m$.

The problem (1.6) - (1.8) has a unique solution in some interval $(0,T_m)$ (Bernier [3]).The following a priori estimates will give us that $T_m = +\infty$. So it admit a solution in any interval (0,T).

(i) We multiply (1.6) by $g_{j,m}(t)$, sum on j and integrate over Ω. Using (1.7) and the Green formula, we find

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\psi_m|^2 + \varepsilon|\nabla\psi_m|^2 + A|\xi_m|^2 - \int_\Omega \frac{\partial \psi_m}{\partial x}\psi_m - \int_\Omega J(\psi_m, \xi_m)\psi_m \le |\nu|\,|\psi_m|$$

thanks to Holder inequality.

Notice that $\int \frac{\partial \psi}{\partial x}\psi = 0$ and $\int J(f,g)f = 0$ when f and g are constants on Γ. Using Poincare inequality, we find :

$$|\nu|\,|\psi_m| \le (\lambda_1)^{1/2}|\nu|\,|\nabla\psi_m| \le \frac{1}{2\lambda_1}|\nu|^2 \inf 1 + \frac{\varepsilon}{2}|\nabla\psi_m|^2 + \frac{A}{2}|\xi_m|^2$$

with $\inf 1 = \inf\left(\frac{1}{\varepsilon}; \frac{1}{A\lambda_1}\right)$ and then

(1.10) $\frac{\partial}{\partial t}|\nabla\psi_m|^2 + \varepsilon\,|\nabla\psi_m|^2 + A\,|\xi_m|^2 \le |\nu|^2 \inf 1/\lambda_1$.

After integration in time and the use of (1.9), it follows that

$$(1.11)\qquad |\nabla\psi_m(t)|^2 + \varepsilon \int_0^t |\nabla\psi_m|^2 + A \int_0^t |\xi_m|^2 \le T\,|v|^2 \inf 1/\lambda_1 + |\nabla\psi(0)|^2 .$$

Using the Poincare inequality in (1.10), we have

$$\frac{\partial}{\partial t} |\nabla\psi_m|^2 + (\varepsilon + A\,\lambda_1)\ |\nabla\psi_m|^2 \le |v|^2 \inf 1/\lambda_1$$

and integrating in time using the Gronwall lemma and (1.9), we obtain

$$(1.12)\qquad |\nabla\psi_m(t)|^2 \le |\nabla\psi(0)|^2 \exp(-\varepsilon_A t) + \frac{|v|^2 \inf 1}{\lambda_1\,\varepsilon_A}\Big(1 - \exp(-\varepsilon_A t)\Big)$$

where $\varepsilon_A = (\varepsilon + A\,\lambda_1)$.

(ii) We multiply (1.6) by $h_{j,m}(t)$, sum on j and integrate over Ω. We have, thanks to the Green formula

$$\frac{1}{2}\frac{\partial}{\partial t} |\xi_m|^2 + \varepsilon|\xi_m|^2 + A|\Delta\xi_m|^2 - \int \frac{\partial\psi_m}{\partial x}\xi_m \le \int v\ \xi_m .$$

Using the Holder inequality, we obtain

$$\int v\ \xi - \int \frac{\partial\psi}{\partial x}\xi \le (\,|v| + |\nabla\psi|\,)\,|\xi| \le (\,|v|^2 + |\nabla\psi|^2\,) \inf 1 + \frac{\varepsilon}{2}|\xi|^2 + \frac{A}{2}|\nabla\xi|^2 .$$

Finally

$$(1.13)\qquad \frac{\partial}{\partial t} |\xi_m|^2 + \varepsilon\,|\xi_m|^2 + A\,|\nabla\xi_m|^2 \le 2\,(\,|v|^2 + |\nabla\psi_m|^2\,) \inf 1 .$$

Using (1.11), we obtain that

$$\frac{\partial}{\partial t} |\xi_m|^2 \le 2 \inf 1\ |\nabla\psi(0)|^2 + 2\,|v|^2\ \inf 1\ (1 + t \inf 1/\lambda_1\,) .$$

We now multiply this inequality by t and integrate by parts. We obtain, thanks to (1.11)

$$(1.14)\qquad t\,|\xi_m(t)|^2 \le K_1\ |\nabla\psi(0)|^2 + K_2$$

with $K_1 = 2 \inf 1\ T + 1/A$ and $K_2 = |v|^2 \inf 1\ T\Big(2 + 2\,T \inf 1/\lambda_1 + 1/(A\,\lambda_1)\Big)$.

Integrating now (1.13) between α and t, and using (1.11) and (1.14), we find two constants K_3 and K_4 such that

$$|\xi_m(t)|^2 + \varepsilon \int_\alpha^t |\xi_m|^2 + A \int_\alpha^t |\nabla\xi_m|^2 \le K_3\,|\nabla\psi(0)|^2 + K_4 .$$

These inequalities show us that, for all constant $T > 0$ and all α such that $0 < \alpha < T$:

ξ_m is bounded independently of m in $L^\infty(0,T,H^{-1}(\Omega)) \cap L^\infty(\alpha,T,L^2(\Omega))$.

ξ_m is bounded independently of m in $L^2(0,T,L^2(\Omega)) \cap L^2(\alpha,T,H_0^1(\Omega))$.

We now shall prove that $\frac{\partial \xi_m}{\partial t}$ is bounded independently of m in $L^2(0,T,H^{-2}(\Omega))$. We consider P_m the orthogonal projector of $L^2(\Omega)$ on V_m. So (1.6) means :

$$\frac{\partial \xi_m}{\partial t} = A\Delta\xi_m - \varepsilon\,\xi_m - P_m\left(\frac{\partial \psi_m}{\partial x}\right) - P_m\left(J(\psi_m, \xi_m)\right) + P_m(v).$$

We have $A\Delta\xi_m - \varepsilon\,\xi_m$, $P_m\left(\frac{\partial \psi_m}{\partial x}\right)$ and v bounded independently of m in $L^2(0,T,H^{-2}(\Omega))$. Using Holder inequality and some properties of the operator J, we have, for u in $\mathscr{C}([0,T], H_0^2(\Omega))$

$$(1.15) \qquad < J(\psi_m, \xi_m)\,,\, u > = \left| \int J(u,\psi_m)\, \xi_m \right| \;\le c\, |\xi_m|^2 \;\; \|u\|_{H^2}.$$

And so $\|J(\psi_m, \xi_m)\|_{H^{-2}} \le c\, |\xi_m|^2$ which is bounded independently of m in $L^2(0,T)$. So $\frac{\partial \xi_m}{\partial t}$ is bounded independently of m in $L^2(0,T,H^{-2}(\Omega))$. Hence there exists a subsequence, still denotes by ξ_m such that

$\xi_m \to \xi$ weakly in $L^2(0,T,L^2(\Omega)) \cap L^2(\alpha,T,H_0^1(\Omega))$,

$\frac{\partial \xi_m}{\partial t} \to \frac{\partial \xi}{\partial t}$ weakly in $L^2(0,T,H^{-2}(\Omega))$.

(α), (β), and (γ) follow from (1.11), (1.12), (1.14).

We now prove : $< J(\psi_m, \xi_m)\,,\, u > \to < J(\psi,\xi),u > \quad \forall\, u \in \mathscr{C}(0,T,H_0^2(\Omega))$.

We have defined for all $\psi_m \in L^2(0,T,H^2(\Omega) \cap H_0^1(\Omega))$.

$< J(\psi_m, \xi_m)\,,\, u > = (J(u,\psi_m),\xi_m)$ where $\xi_m = \Delta\psi_m$.

As $\xi_m \to \xi$ weakly in $L^2(0,T,L^2(\Omega))$, $\psi_m \to \psi$ in $L^2(0,T,H_0^1(\Omega))$. Since $J(u,\psi_m) \to J(u,\psi)$ in $L^2(0,T,L^2(\Omega))$, so $(J(u,\psi_m),\xi_m) \to (J(u,\psi),\xi)$. Passing to the limit in (1.6)-(1.8), we find

$$\frac{\partial \xi}{\partial t} = A\Delta\xi - \varepsilon\,\xi - \frac{\partial \psi}{\partial x} - J(\psi, \xi) + v \;\text{ in } L^2(0,T,H^{-2}(\Omega)).$$

Since $\xi \in L^2(0,T,L^2(\Omega))$ and $\frac{\partial \xi}{\partial t} \in L^2(0,T,H^{-2}(\Omega))$, $\xi \in \mathscr{C}(0,T,H^{-1}(\Omega))$. Hence $\xi(.,0)$ makes sense and (1.3) follows from (1.9).

<u>Proof of 3). Continuous dependence on the initial value.</u>

Let (ψ,ξ) and (ψ^*,ξ^*) be two solutions of the problem (1.1) - (1.4) corresponding to the initial values ξ_0 and ξ^*_0 in $H^{-1}(\Omega)$. Then $w = \xi - \xi^*$ and $\Phi = \psi - \psi^*$ satisfy

$$(1.16) \qquad \frac{\partial w}{\partial t} + \varepsilon\, w + \frac{\partial \Phi}{\partial x} - A\,\Delta w + J(\psi,\xi) - J(\psi^*,\xi^*) = 0$$

where $\Delta\Phi = w$, $\;\Phi = 0$ on $\partial\Omega$.

(1.17) $\qquad w(.,0) = \xi_0(.) - \xi^*_0(.).$

We notice that $J(\psi,\xi) - J(\psi^*,\xi^*) = J(\Phi,\xi) + J(\psi^*,w)$.

Multiplying (1.16) by Φ, and integrating on Ω, we obtain that

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\Phi|^2 + \varepsilon|\nabla\Phi|^2 + A|w|^2 \leq |\langle J(\psi^*,w),\Phi\rangle| \quad \text{for } t>0$$

thanks to the Green formula. An integration by parts gives us that

$$|\langle J(\psi^*,w),\Phi\rangle| \leq c\,|\nabla\Phi|^2_{L^4}\,|\Delta\psi^*|\,.$$

We use the Holder inequality and we denote by c any constant depending on Ω.

$$|\langle J(\psi^*,w),\Phi\rangle| \leq c\,|\xi^*|\,|\nabla\Phi|^{\sigma}_{L^2}\,|\Delta\Phi|^{2-\sigma}_{L^2}\ .$$

Thanks to the Young inequality, we obtain

$$|\langle J(\psi^*,w),\Phi\rangle| \leq c\,|\xi^*|\,|\nabla\Phi|^2_{L^2} + A\,|\Delta\Phi|^2_{L^2}\ .$$

So

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\Phi|^2 + \varepsilon|\nabla\Phi|^2 + A|w|^2 \leq c\,|\xi^*|\,|\nabla\Phi|^2 + A|w|^2\ .$$

Then

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\Phi|^2 \leq c\,|\xi^*|\,|\nabla\Phi|^2.$$

Whence, integrating in time between α and t, $0 < \alpha \leq t \leq T$

$$|\nabla\Phi(t)|^2 \leq |\nabla\Phi(\alpha)|^2 \exp(2\,c\,K)$$

where $K \geq \int_0^T |\xi^*|$. Since $|\nabla\Phi(t)| \in \mathcal{C}([0,T])$, we obtain

(1.18) $\qquad |\nabla\Phi(t)|^2 \leq |\nabla\Phi(0)|^2 \exp(2\,c\,K)\;$ for $0 \leq t \leq T$.

This result gives us the continuous dependance on the initial value and the uniqueness of the solution. The proof of theorem 1.1 is completed.

PART 2 : THE MAXIMAL ATTRACTOR.

2.1. Existence.

In this section, we first recall general definitions and a sufficient condition ensuring the existence of an attractor (Marion [12], Temam [14]). We then prove the existence of absorbing sets and of a maximal attractor for the problem (1.1) - (1.4) .

2.1.1. General results and definitions.

Let H be a metric space and let $S(t)$, $t \geq 0$ be a semi-group of operators from H into itself.

Definition 2.1. The ω-limit set of subset $\mathcal{C}$ of H is defined by :

$$\omega(\mathcal{C}) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t)\mathcal{C}}^{H} .$$

Definition 2.2. A functional invariant set for the semi-group $S(t)$ is a subset $\mathcal{J}$ of H such that $S(t)\, \mathcal{J} = \mathcal{J}$ for all $t > 0$.

A functional invariant set $\mathcal{A}$ is said to be an attractor, if it possesses an open neighborhood $\mathcal{U}$ such that, for every u in $\mathcal{U}$

$$\operatorname{dist}(S(t)\, u, \mathcal{A}) \rightarrow 0 \text{ as } t \rightarrow +\infty .$$

The largest open set which contains $\mathcal{A}$ and enjoys the same property as $\mathcal{U}$ is called the basin of attraction of $\mathcal{A}$.

Definition 2.3. A subset $\mathcal{B}$ of H is said to be absorbing in H for the semi-group $S(t)$ if, for every bounded set B_0 of H, there exists $T = T(B_0)$ such that $S(t)\, B_0 \subset \mathcal{B}$ for all $t \geq T(B_0)$.

We assume that

(2.1) $S(t)$ is continuous from H into itself for all $t \geq 0$.

(2.2) The operators $S(t)$ are uniformly compact in the following sense : for every bounded set B, there exist $t_0 \geq 0$ such that

$$\bigcup_{t \geq t_0} S(t)\, B \text{ is relatively compact in H} .$$

Theorem 2.4.(Temam [14]) Under the assumptions (2.1) and (2.2), and if there exists a bounded absorbing set $\mathcal{B}$, then the ω-limit set of $\mathcal{B}$: $\mathcal{A} = \omega(\mathcal{B})$ is a compact attractor which attracts the bounded sets of H. It is the maximal bounded attractor in H and if H is a convex subset in a Banach space, then $\mathcal{A}$ is connected.

2.1.2. Existence of absorbing sets and of a maximal attractor.

Theorem 2.5. The semi-group $S(t)$ from $H^{-1}(\Omega)$ to $H^{-1}(\Omega)$: $\xi(.,0) \rightarrow \xi(.,t)$ associated to (1.1) -(1.4) is such that :

(i) there exist absorbing sets in $H^{-1}(\Omega)$, bounded in $H^{-1}(\Omega)$,

(ii) $S(t)$ is $(H^{-1}(\Omega), L^2(\Omega))$ - bounded for $t > 0$, ie
$S(t)B \in \mathcal{B}(L^2(\Omega)), \forall B \in \mathcal{B}(H^{-1}(\Omega))\ t > 0$

with $\mathcal{B}(V) = \{\text{set } B \text{ of } H^{-1}(\Omega), B \text{ bounded in } V\}$,

(iii) there exists a maximal attractor $\mathcal{A}$ which is bounded in $L^2(\Omega)$, compact and connected in $H^{-1}(\Omega)$. Its basin of attraction is the whole space $H^{-1}(\Omega)$.

Proof of theorem 2.5.

(i) Absorbing set in $H^{-1}(\Omega)$.

In section 1, we obtained the inequality (β) that we recall

$$|\nabla\psi(t)|^2 \le |\nabla\psi(0)|^2 \exp(-\varepsilon_A t) + \frac{|v|^2 \inf l}{\lambda_1 \varepsilon_A}(1-\exp(-\varepsilon_A t)) \tag{2.3}$$

We deduce from (2.3) that any ball of $H^{-1}(\Omega)$ centered at 0 of radius

$$r_2 > r_1 = |v| \sqrt{\frac{\inf l}{\lambda_1 \varepsilon_A}} \tag{2.4}$$

is an absorbing set in $H^{-1}(\Omega)$. Indeed, if $\mathcal{B}$ is a bounded set of $H^{-1}(\Omega)$ included in a ball $B(0,R)$ of $H^{-1}(\Omega)$ centered at 0 of radius R, then $S(t)\ \mathcal{B} \subset B(0,r_2)$ for $t \ge T_0 = T_0(\mathcal{B}, r_2)$

$$T_0 = \frac{1}{\varepsilon} \operatorname{Log}\left(\frac{R^2}{r_2{}^2 - r_1{}^2}\right).$$

(ii) $S(t)$ is $(H^{-1}(\Omega), L^2(\Omega))$ -bounded for $t > 0$.

This result follows from the inequality (γ) obtained in part 1

$$t\,|\xi(t)|^2 \le K_1\, |\nabla\psi(0)|^2 + K_2.$$

(iii) Absorbing set in $L^2(\Omega)$.

Let $r_2 > r_1$ be fixed. We infer from (1.10) after integrating in time that, for $r > 0$ and $t \ge 0$,

$$\varepsilon \int_t^{t+r} |\nabla\psi(s)|^2\, ds + A \int_t^{t+r} |\xi(s)|^2\, ds \le r \inf l\ |v|^2 / \lambda_1 + |\nabla\psi(t)|^2. \tag{2.5}$$

Whence, for $\xi_0 \in \mathcal{B} \subset B(0,R)$ and $t \ge T_0$, $\forall\, r > 0$,

$$\varepsilon \int_t^{t+r} |\nabla\psi(s)|^2\, ds + A \int_t^{t+r} |\xi(s)|^2\, ds \le r \inf l\ |v|^2 / \lambda_1 + r_2^2. \tag{2.6}$$

We will use this inequality later.

In section 1, we obtain the inequality (1.13) which is available for $t > 0$ thanks to (ii),

$$\frac{\partial}{\partial t}\,|\xi|^2 + \varepsilon\,|\xi|^2 + A\,|\nabla\xi|^2 \le 2\,(|v|^2 + |\nabla\psi|^2)\,\mathrm{infl}$$

we rewrite

$$\frac{\partial}{\partial t}\,|\xi|^2 \le 2\,(|v|^2 + |\nabla\psi|^2)\,\mathrm{infl} \tag{2.7}$$

and recall the uniform Gronwall lemma :

Lemma 2.6. Let g, h, y three locally integrable functions on $]\,t_0, +\infty[$, which satisfy

$$\frac{\partial y}{\partial t} \in L^1_{loc}\,(\,]0, +\infty[\,) \text{ and } \frac{\partial y}{\partial t} \le g\,y + h \text{ for } t \ge t_0.$$

$$\int_t^{t+r} g(s)\,ds \le a_1\,,\quad \int_t^{t+r} h(s)\,ds \le a_2,\quad \int_t^{t+r} y(s)\,ds \le a_3 \quad \text{for } t \ge t_0.$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \le \left(\frac{a_3}{r} + a_2\right)\exp\,(a_1) \tag{2.8}$$

The proof of this lemma can be find in Foias-Manley-Temam [7].

Let $r > 0$ be fixed, $y = |\xi|^2$, $g = 0$, $h = \dfrac{2}{\varepsilon}\,(|v|^2 + |\nabla\psi|^2\,)$. We infer from (2.6) that, for $\xi_0 \in \mathcal{B} \subset B(0,R)$ and $t \ge T_0$

$$a_1 = 0\,,\qquad a_2 = 2\,\mathrm{infl}\;r\,(\,|\,v\,|^2 + r_2^2\,)\,,\qquad a_3 = (r\,c_1^2\,|\,v\,|^2\,\mathrm{infl} + r_2^2\,)/A.$$

We can apply Lemma 2.6 to (2.7). We derive from (2.8) the existence of a constant $r_3^2 > 0$ such that, if $\xi_0 \in \mathcal{B} \subset B(0,R)$ and for $t \ge T_0 + r$

$$|\xi|^2 \le r_3^2. \tag{2.9}$$

We obtain from (2.9) that the ball $\mathcal{B}_1$ of $L^2(\Omega)$, centered at 0 of radius r_3 is absorbing in $L^2\,(\Omega)$. At the same time, this result provides the uniform compactness of $S(t)$ in $H^{-1}(\Omega)$. If $\xi_0 \in \mathcal{B} \subset B(0,R)$ and $t \ge T_0 + r$, $\xi(.,t)$ belongs to $\mathcal{B}_1$ which is bounded in $L^2(\Omega)$ and relatively compact in $H^{-1}\,(\Omega)$.

To conclude, the assumptions (2.1) and (2.2) are satisfied and we have proved the existence of a bounded absorbing set in $H^{-1}\,(\Omega)$. Hence, theorem 2.4 applies and gives theorem 2.5 (iii).

2.2 Dimension of the maximal attractor.

Our aim is now to estimate in terms of the data the dimension of the attractor introduced before. We start by recalling a few results (Constantin-Foias-Temam [6], Temam [14], Marion [12]). We then derive estimates of the dimension of the attractor.

2.2.1. General results.

Let H be an Hilbert space (norm $|\ |$) and $\mathcal{X}$ be a functional invariant set compact in H for a semi-group $(S(t))_{t\geq 0}$.

We assume that for all $t > 0$,

(2.10) $S(t)$ is uniformly differentiable in $\mathcal{X}$, i.e. for every $u \in \mathcal{X}$, there exists a linear operator in H, $L = L(t,u)$ in $\mathcal{L}(H)$ such that

$$\sup_{\substack{u,v \in \mathcal{X} \\ 0 < |v-u| < \varepsilon}} \frac{|S(t)\, v - S(t)u - L(t,u)\,(v-u)|}{|v - u|} \to 0 \text{ as } \varepsilon \to 0 .$$

(2.11) $$\sup_{u \in \mathcal{X}} |L(t,u)|_{\mathcal{L}(H)} < +\infty .$$

For all $L \in \mathcal{L}(H)$ and $N \in \mathbf{N}^*$, we denote by $\omega_N(L)$ the norm of exterior product $\Lambda^N L$ in $\Lambda^N H$:

(2.12) $$\omega_N(L) = \sup_{\substack{\xi_1,\dots,\xi_N \in H \\ |\xi_i| \leq 1}} |L\xi_1 \wedge \dots \wedge L\xi_N| .$$

We introduce $\bar{\omega}_N(t) = \sup_{u_0 \in \mathcal{X}} \omega_N(L(t,u_0))$ for $t \geq 0$, $N \in \mathbf{N}^*$

and $\Pi_N = \lim_{t \to +\infty} \bar{\omega}_N(t)^{1/t}$. This limit exists since the mapping $t \to \bar{\omega}_N(t)$ is subexponential.

We define the uniform Lyapunov numbers μ_i by

$$\mu_1 = \operatorname{Log} \Pi_1 \text{ and } \mu_N = \operatorname{Log} \Pi_N - \operatorname{Log} \Pi_{N-1} \text{ for } N \geq 2 .$$

Theorem 2.7. Under assumptions (2.10) - (2.11), if for some integer $N \geq 1$, $\mu_1 + \dots + \mu_N < 0$, then the Hausdorff dimension of $\mathcal{X}$ is less than or equal to N and the fractal dimension of $\mathcal{X}$ is less than or equal to

(2.13) $$N \max_{1 \leq l < N} \left\{ 1 + \frac{(\mu_1 + \dots + \mu_l)_+}{|\mu_1 + \dots + \mu_N|} \right\}$$

(Constantin-Foias-Temam [6]).

Let us recall the definition of the Hausdorff and fractal dimensions : the <u>d-dimensional Hausdorff measure</u> of $\mathcal{J}$ is the number

$$(2.14) \qquad \mu_H(\mathcal{J},d) = \lim_{\varepsilon \to 0} \mu_H(\mathcal{J},d,\varepsilon)$$

where

$$\mu_H(\mathcal{J},d,\varepsilon) = \inf\left(\sum_i r_i^d\right)$$

the infimum being taken for all the covering of $\mathcal{J}$ by balls of radii $r_i \leq \varepsilon$. There exists $d_H(\mathcal{J})$ such that $\mu_H(\mathcal{J},d) = 0$ for $d > d_H(\mathcal{J})$ and $= +\infty$ for $d < d_H(\mathcal{J})$. $d_H(\mathcal{J})$ is the <u>Hausdorff dimension</u> of $\mathcal{J}$.

The <u>fractal dimension</u> of $\mathcal{J}$ is

$$(2.15) \qquad d_F(\mathcal{J}) = \inf\left\{ d > 0,\ \mu_F(\mathcal{J},d) = 0 \right\}$$

where

$$\mu_F(\mathcal{J},d) = \lim_{\varepsilon \to 0} \sup \varepsilon^d \eta_{\mathcal{J}}(\varepsilon)$$

and $\eta_{\mathcal{J}}(\varepsilon)$ is the minimum numbers of balls of radius ε which is necessary to cover $\mathcal{J}$. It is clear that the fractal dimension of a set is larger than or equal to its Hausdorff dimension.

2.2.2 Estimate of the dimension of the attractor.

We now intend to apply Theorem 2.7 to the maximal attractor defined in theorem 2.5.

<u>Theorem 2.8.</u> The fractal and Hausdorff dimensions of the maximal attractor $\mathcal{A}$ defined in theorem 2.5 are finite.

<u>Proof :</u> We need the following lemma (the proof is in Appendix A) :

<u>Lemma 2.9.</u> For every $t_0 > 0$, $S(t_0)$ is uniformly differentiable on $\mathcal{A}$. Its differential at ξ_0 is the linear operator on $H^{-1}(\Omega)$: $U_0 \to L(t_0, \xi_0)\, U_0 = U(t_0)$ where $U(t_0)$ is the value at time $t = t_0$ of the solution $U(t)$ of the linearized problem :

$$(2.16) \qquad \frac{\partial U}{\partial t} = -\varepsilon U - \frac{\partial V}{\partial x} + A\,\Delta U - J(\psi, U) - J(V, \Delta\psi) \qquad \text{in } \Omega \times \mathbf{R}^+$$

$$(2.17) \qquad \Delta V = U \qquad \text{in } \Omega \times \mathbf{R}^+$$

$$(2.18) \qquad \Delta V = 0 \text{ and } V = 0 \qquad \text{on } \Gamma$$

$$(2.19) \qquad U(t = 0) = U_0$$

where (ψ,ξ) is the solution of (1.1) - (1.4). Moreover, we have

$$(2.20) \qquad \sup_{\xi_0 \in \mathcal{A}} |\, L(t_0,\xi_0)\, |_{\mathcal{L}(H^{-1})} < +\infty .$$

We now intend to estimate $\mu_1 + \ldots + \mu_n = \operatorname{Log} \Pi_n$ for $n \in \mathbb{N}$. We introduce

$$F'(\xi)U = -\varepsilon U - \frac{\partial V}{\partial x} + A\,\Delta U - J(\psi,U) - J(V,\Delta\psi)$$

where $\Delta V = U$, $V \in H_0^1(\Omega)$ and (ψ, ξ) is the solution of (1.1) - (1.4). According to definition (2.12), we have

$$\omega_N(L(t,\xi_0)) = \sup_{\substack{U_j^0 \in H^{-1}(\Omega) \\ |U_j^0| \leq 1}} |U_1(t) \wedge \ldots \wedge U_N(t)|$$

where U_j satisfies

$$\frac{\partial U_j}{\partial t} = F'(\xi)\, U_j$$

$$U_j(.,0) = U_j^0(.)$$

It can be shown (Constantin-Foias-Temam [6], Temam [14]) that

$$\omega_N(L(t,\xi_0)) \leq \sup_{\substack{1 \leq j \leq N \\ |U_j^0| \leq 1}} \left(\exp \int_0^t Tr(F'(\xi(s)) o Q^N(s))\, ds \right)$$

where $Q^N(t)$ denotes the orthogonal projector in $H^{-1}(\Omega)$ onto the space spanned by $\{U_1(t), \ldots, U_N(t)\}$. We introduce

$$q^N(t) = \sup_{\xi_0 \in \mathcal{A}} \sup_{\substack{U_j^0 \in H^{-1} \\ |U_j^0| \leq 1}} \left(\frac{1}{t} \int_0^t Tr(F'(\xi(s)) o Q^N(s))\, ds \right)$$

and

$$q_N = \limsup_{t \to +\infty} q^N(t)$$

and we obtain that

$$\Pi_N \leq \exp(q_N).$$

Then, if $q_N < 0$ for some N, the Hausdorff dimension of $\mathcal{A}$ is less than or equal to N and the fractal dimension is majorised by

(2.13) $$N \max_{1 \leq l \leq N} \left\{ 1 + \frac{(-q_l)_+}{|q_N|} \right\}.$$

So, let us consider m solutions $U_1, ..., U_m$ of (2.16) - (2.18) corresponding to m initial conditions $U_1^0, ..., U_m^0$. We introduce $Q_m(\tau) = Q_m(\tau, \xi_0, U_1^0, ..., U_m^0)$ the orthogonal projector in $H^{-1}(\Omega)$ onto the space spanned by $U_1(\tau), ..., U_m(\tau)$.

We fixe τ. We consider $\phi_j(\tau)$, $j = 1, ..., m$ an orthogonal basis of $Q_m(\tau)H^{-1}(\Omega)$ for the new scalar product $\langle\langle\, .\, ,\, .\,\rangle\rangle$. We denote by $\psi_j(\tau)$ the functions defined by $\Delta\psi_j(\tau) = \phi_j(\tau)$ in Ω, $\psi_j(\tau) = 0$ on Γ. We now try to estimate

$$\mathrm{Tr}(F'(S(\tau)\xi_0) o Q_m(\tau)) = \sum_{j=1}^{m} \langle\langle F'(\xi(\tau))\phi_j(\tau), \phi_j(\tau)\rangle\rangle \ .$$

We have

$$\langle\langle F'(\xi)\phi_j, \phi_j\rangle\rangle = -\varepsilon \langle\langle \phi_j, \phi_j\rangle\rangle - \langle\langle \frac{\partial\psi_j}{\partial x}, \phi_j\rangle\rangle + A \langle\langle \Delta\phi_j, \phi_j\rangle\rangle$$

$$- \langle\langle J(\psi, \phi_j), \phi_j\rangle\rangle - \langle\langle J(\psi_j, \xi), \phi_j\rangle\rangle \ .$$

By definition of $\langle\langle\, ,\,\rangle\rangle$, we obtain

$$\langle\langle F'(\xi)\phi_j, \phi_j\rangle\rangle = -\varepsilon |\nabla\psi_j|^2 + (\frac{\partial\psi_j}{\partial x}, \psi_j) - A|\Delta\psi_j|^2 + \langle J(\psi, \phi_j), \psi_j\rangle + \langle J(\psi_j, \xi), \psi_j\rangle \ .$$

$\langle\langle F'(\xi)\phi_j, \phi_j\rangle\rangle = -\varepsilon |\nabla\psi_j|^2 - A|\Delta\psi_j|^2 + \langle J(\psi_j, \psi), \Delta\psi_j\rangle$ as in part (1.1.2).

Integrating by parts, we have

$$\langle J(\psi_j, \psi), \Delta\psi_j\rangle = -\int \frac{\partial^2\psi}{\partial x\partial y}\left((\frac{\partial\psi_j}{\partial x})^2 - (\frac{\partial\psi_j}{\partial y})^2\right)d\omega + \int \frac{\partial\psi_j}{\partial x}\frac{\partial\psi_j}{\partial y}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)d\omega \ .$$

We introduce $\rho_j(x,y) = \left(\frac{\partial\psi_j}{\partial x}\right)^2 + \left(\frac{\partial\psi_j}{\partial y}\right)^2$, and $\rho(x,y) = \sum_{j=1}^{m} \rho_j(x,y)$ and then

$$\sum_{j=1}^{m} \langle J(\psi_j, \psi), \Delta\psi_j\rangle \leq \int \left|\frac{\partial^2\psi}{\partial x\partial y}\right| \rho(x,y)\, d\omega + \frac{1}{2}\int \rho(x,y)\left(\left|\frac{\partial^2\psi}{\partial x^2}\right| + \left|\frac{\partial^2\psi}{\partial y^2}\right|\right)d\omega \ ,$$

Using Schwartz inequality, we obtain

$$(2.21) \qquad \sum_{j=1}^{m} \langle J(\psi_j, \psi), \Delta\psi_j\rangle \leq \|y\|_H^2\ |\rho| \ .$$

Therefore

$$(2.22) \qquad \sum_{j=1}^{m} \langle\langle F'(\xi)\,\phi_j, \phi_j\rangle\rangle \leq -\varepsilon \sum_{j=1}^{m} |\nabla\psi_j|^2 - A\sum_{j=1}^{m} |\Delta\psi_j|^2 + \|\psi\|_H^2\,|\rho| \ .$$

Let us introduce $u_j = \begin{pmatrix} \frac{\partial \psi_j}{\partial x} \\ \frac{\partial \psi_j}{\partial y} \end{pmatrix}$ $j = 1, \ldots, m$. As $\{\psi_j\}$ is an orthonormal **family for the** scalar product $<< , >>$, $\{u_j\}$ is an orthonormal family in $(L^2(\Omega))^2$. Using the Sobolev-Lieb-Thirring inequality (Temam [14]), we obtain that there exists a constant $c(\Omega)$ such that :

$$|\rho|^2 \leq c^2(\Omega) \sum_{j=1}^{m} |\nabla u_j|^2 .$$

Using the Youdovitch inequality, we have

$$|\rho|^2 \leq c^2 c_y^2 \sum_{j=1}^{m} |\Delta \psi_j|^2 . \tag{2.23}$$

Combining (2.22) and (2.23)

$$\sum_{j=1}^{m} << F'(\xi)\phi_j, \phi_j >> \leq - \varepsilon \sum_{j=1}^{m} |\nabla \psi_j|^2 - A \sum_{j=1}^{m} |\Delta \psi_j|^2 +$$
$$+ c\, c_y \|\psi\|_{H^2}^2 \left(\sum_{j=1}^{m} |\Delta \psi_j|^2 \right)^{1/2} .$$

$$\sum_{j=1}^{m} << F'(\xi)\phi_j, \phi_j >> \leq - \varepsilon \sum_{j=1}^{m} |\nabla \psi_j|^2 - \frac{A}{2} \sum_{j=1}^{m} |\Delta \psi_j|^2 + \frac{c^2 c_y^2}{2A} \|\psi\|_{H^2}^2 .$$

As $\{\phi_j\}$ is orthonormal for $<< , >>$, we have

$$\sum_{j=1}^{m} |\nabla \psi_j|^2 = m$$

and, since the eigenvalues of $-\Delta$ satisfy $\lambda_j \sim c\lambda_1 j$ as $j \to \infty$, we obtain (Teman [14])

$$\sum_{j=1}^{m} |\Delta \psi_j|^2 \geq c_e \lambda_1 m^2 .$$

Then

$$\sum_{j=1}^{m} << F'(\xi)\phi_j, \phi_j >> \leq - \varepsilon m - c_e \lambda_1 \frac{A}{2} m^2 + \frac{c^2 c_y^2}{2A} \|\psi\|_{H^2}^2 .$$

We deduce that

$$q_m(t) \leq -\frac{A}{2}\, c_e \lambda_1 m^2 + \frac{c^2 c_y^2}{2A} \sup_{\xi_0 \in \mathcal{A}} \left(\frac{1}{t} \int_0^t \|\psi(\tau)\|_{H^2}^2 \, d\tau \right).$$

Using (α), it follows that

$$\left(\frac{1}{t} \int_0^t |\Delta\psi(\tau)|^2 \, d\tau \right) \leq \frac{|v|^2}{A\lambda_1} \inf 1 + \frac{1}{t} \frac{|\nabla\psi(0)|^2}{A} . \tag{2.24}$$

So we obtain

$$\sup_{\xi_0 \in \mathcal{A}} \left(\frac{1}{t} \int_0^t \|\psi(\tau)\|_{H^2}^2 \, d\tau \right) \leq \frac{|v|^2 c_y}{A\lambda_1} \inf 1 + \frac{c_y}{A} \sup_{\xi_0 \in \mathcal{A}} \frac{1}{t} |\nabla\psi(0)|^2$$

and finally

$$q_m \leq \lim_{t \to +\infty} q_m(t) \leq -\frac{A}{2} c_e \lambda_1 m^2 + \frac{|v|^2 c^2 c_y^3}{2A^2 \lambda_1} \inf 1 = -K_5 m^2 + K_6.$$

Let now m' be the integer such that

$$m' - 1 < \sqrt{\left(\frac{2K_6}{K_5}\right)} \leq m' .$$

Then $q_{m'} < 0$. As observe before, the Hausdorff dimension is majorized by m' and (2.13) gives us that the fractal dimension is majorized by m' $(1 + K_6/K_6) = 2 * m'$.
The expression of the bound of m' is :

$$m' < 1 + \sqrt{\left(\frac{2K_6}{K_5}\right)}$$

$$m' < 1 + \left(\frac{2c_y^3 |v|^2 c^2 \inf 1}{A^3 \lambda_1^2 c_e} \right)^{1/2} = 1 + C \left(\frac{|v|^2 \inf 1}{A^3} \right)^{1/2}$$

with $\inf 1 = \inf(\frac{1}{\varepsilon} ; \frac{1}{A\lambda_1})$ and C depend on Ω.

THE MULTILAYER PROBLEM.

In these parts, we consider the baroclinic problem, when the ocean is divided in layers. For each layer k of thickness H_k, we have two unknown functions : the stream function ψ_k and the vorticity ξ_k. The layer k is coupled with the layers k + 1 and k - 1 by a non linear term. The energy is given by the curl of the wind on the upper layer and dissipated by bottom and lateral frictions (Chow-Holland [5], Le Provost [9]). We first prove the existence and uniqueness of the solution and then we study the existence of a maximal attractor and estimate its Hausdorff and fractal dimensions.

PART 3. EXISTENCE AND UNIQUENESS OF THE SOLUTION.

3.1 Equations and theorem.

Let Ω denote a regular enough open bounded set of $\mathbb{R}^2$ with boundary Γ. We consider the following initial boundary value problem involving scalar funcfions ξ_k, $k = 1, ..., N$, with $N \in \mathbb{N}^*$.

$$(3.1_k)\quad \frac{\partial \xi_k}{\partial t} + J(\psi_k, \xi_k + f) + \frac{f_0}{H_k}(w_k - w_{k-1}) - A\,\Delta\xi_k = S_k + F_k \quad \text{on } \Omega x(0,T),\ k=1...N$$

$$(3.2_k)\quad w_k = \frac{f_0}{g_k}\left(\frac{\partial}{\partial t}(\psi_{k+1} - \psi_k) + J(\psi_k, \psi_{k+1})\right) \qquad \text{on } \Omega x(0,T),\quad k=1...N-1$$

$$(3.3_k)\quad \Delta\psi_k = \xi_k \qquad \text{on } \Omega x(0,T),\quad k=1...N$$

where the unknown functions are the vorticity ξ_k and the stream function ψ_k. The layer k is caracterized by its thickness H_k and by its reduced gravity g_k. The forcing S infers only on the upper layer : $S_k = 0$ for $k \neq 1$ and $S_1 = \frac{v}{H_1}$ where v is the curl of the wind and is supposed independent of time and in $L^2(\Omega)$. The energy is dissipated in each layer by the lateral friction $A\,\Delta\xi_k$, and for the bottom layer by the bottom friction $F_N = -\varepsilon\,\xi^N$ ($F_k = 0$ for $k \neq N$). All the constants A, ε, etc are positive. We consider the linear approximation of the parameter of Coriolis f : $f = f_0 + \beta_0 y$ with f_0 the parameter of Coriolis in the middle of the ocean, β_0 a physical constant. The layer k is coupled with the layer k+1 [resp. k-1] by the term w_k [resp. w_{k-1}] which represents the vertical velocity of the interface between the layers k and k+1 [resp. k-1 and k]. For simplicity, we introduce the terms w_0 and w_N which are equal to zero. The non-linear terms are due to the Jacobian operator.

The initial condition is

$$(3.4_k)\qquad \xi_k(.,0) = \xi_k^0(.) \qquad \xi_k^0 \text{ is given in } H^{-1}(\Omega),$$

$$(3.5)\qquad \int_\Omega \psi(.,0)\, d\omega = 0 \qquad \text{where } \Delta\psi(.,0) = \Delta\psi^0(.) = \xi^0(.).$$

The boundary conditions are the following

(3.6_k) $\Delta\psi_k = 0$ on $\partial\Omega$,

(3.7_k) $\psi_k(.,t) = C_k(t)$ on $\partial\Omega$, $C_k(t) \in \mathbf{R}$.

For each layer, we write the mass conservative law

$$(3.8_k) \qquad \int_\Omega w_k \, d\omega = 0 \qquad \text{for } 1 \le k \le N.$$

We note by ψ the vector ${}^t(\psi_1, ..., \psi_N)$, and by H_0^1 the space $(H_0^1(\Omega))^N$, $H^2 \cap H_0^1 = (H^2(\Omega) \cap H_0^1(\Omega))^N$, etc and introduce the NxN matrix W defined by

$$W_{k,j} = 0 \text{ if } |k - j| > 1$$

$$(3.9) \qquad W_{k,k} = \frac{f_0^2}{H_k}\left(\frac{1}{g_k} + \frac{1}{g_{k-1}}\right) \; ; \; W_{k,k+1} = \frac{-f_0^2}{H_k g_k} \; ; \; W_{k,k-1} = \frac{-f_0^2}{H_k g_{k-1}} .$$

We rewrite the equation on the following form :

$$(3.10) \qquad \frac{\partial\theta}{\partial t} = \big(J(\psi_k, \theta_k + f)\big) + A\,\Delta^2\psi + S + F$$

$$(3.11) \qquad \Delta\psi - W\psi = \theta$$

$$(3.12) \qquad \theta(.,0) = \theta^0(.)$$

$$(3.13) \qquad \int_\Omega \psi(.,0)\, d\omega = 0$$

$$(3.14) \qquad \Delta\psi = 0 \qquad \text{on } \partial\Omega$$

$$(3.15) \qquad \psi(.,t) = C(t) \qquad \text{on } \partial\Omega$$

(3.5)-(3.8) give us

$$(3.16) \qquad \int_\Omega \psi_1 \, d\omega = \int_\Omega \psi_k \, d\omega = ... = \int_\Omega \psi_N \, d\omega \qquad \text{for } 1 \le k \le N.$$

Obviously, the two problems (3.1) - (3.8) and (3.10) - (3.16) are formally equivalent.

Let now Λ be the diagonal matrix of eigenvalues of W, and $\mathcal{B}$ and $\mathcal{B}^{-1}$ the two matrices such that $\Lambda = \mathcal{B}^{-1}\, W\, \mathcal{B}$. The matrix W has positive or nul eigenvalues. We introduce $\psi^M = \mathcal{B}^{-1}\psi$, $C^M(t) = \mathcal{B}^{-1}C(t)$ etc...We normelize ψ by the condition $C_1^M = 0$ which is equivalent to (3.17)

$$(3.17) \qquad \sum_k b_{1,k}^{-1}\, C_k(t) = 0$$

where the coefficient $b_{i,j}^{-1}$ are the coefficients of the matrix $\mathcal{B}^{-1}$. Notice that the system (3.11), (3.15) and (3.16) is equivalent to the system

$$\begin{aligned}
&\Lambda\psi^M - \Delta\psi^M = \theta^M && \text{in } \Omega.\\
&\int_\Omega \psi_k^M \, d\omega = 0 && \text{for } k = 2,\dots,N\\
&\psi^M = C^M && \text{on } \partial\Omega
\end{aligned}$$

which admits a unique solution in H^1 for θ^M in H^{-1}.

Let us introduce

(3.18) $\psi' \in H^1_0$ such that $\psi = \psi' + C$.

(3.19) $H^1_c = \{\psi' + C, \psi' \in H^1_0, C \in (\mathbf{R}^N)^2, C_1^M = 0, \int_\Omega \psi_1 \, d\omega = \int_\Omega \psi_k \, d\omega \text{ for } 1 \le k \le N\}$

For θ and $\tilde\theta$ in H^{-1}, we introduce $(\psi,\tilde\psi) \in (H^1_c)^2$, $(\psi',\tilde\psi') \in (H^1_0)^2$ and $(C, \tilde C) \in (\mathbf{R}^N)^2$, such that

$$\psi = \psi' + C, \qquad \tilde\psi = \tilde\psi' + \tilde C$$

and

$$\theta = \Delta\psi - W\psi = \Delta\psi - W\psi' - WC \qquad \tilde\theta = \Delta\tilde\psi - W\tilde\psi = \Delta\tilde\psi - W\tilde\psi' - W\tilde C$$

We equip H^{-1} with the scalar product

$$\langle\langle\theta, \tilde\theta\rangle\rangle = -\langle \theta, (H\tilde\psi')\rangle$$

with the notation $(H\tilde\psi')_k = H_k\tilde\psi'_k$. So

$$\langle\langle\theta, \tilde\theta\rangle\rangle = \sum_k H_k \int \nabla\psi_k.\nabla\tilde\psi_k + \sum_k p_k \int (\psi_k - \psi_{k+1}).(\tilde\psi'_k - \tilde\psi'_{k+1})$$

where $p_k = f_0^2 / g_k$. Thanks to (1.16), we have

(1.20) $$\langle\langle\theta, \tilde\theta\rangle\rangle = \sum_k H_k \int \nabla\psi_k.\nabla\tilde\psi_k + \sum_k p_k \int (\psi_k - \psi_{k+1}).(\tilde\psi_k - \tilde\psi_{k+1}).$$

We denote by $|\theta|_{-1}$ the norm $|\theta|^2_{-1} = \langle\langle\theta,\theta\rangle\rangle$ which is equivalent to the usual norm on $H^{-1}(\Omega)$. We equip L^2 with the norm $\| \,.\, \|_2$ defined by

(1.21) $$\|\theta\|_2^2 = \sum_k H_k |\xi_k|^2 + \sum_k p_k |\nabla\psi_{k+1} - \nabla\psi_k|^2$$

It is easy to prove that $\|\theta\|_2^2 \ge c(\Omega)\, |\theta|_2^2$.

<u>Theorem 1.1.</u> 1) The problem (3.10) - (3.17) admits a unique solution $(\theta_k)_{k=1,\dots,N} = \theta$

$$\theta \in \mathscr{C}([0,T], H^{-1}) \cap L^2(0,T,L^2) \cap L^2_{loc}(]0,T], H^1).$$

2) This solution verifies for $0 \le t \le T$

(3.α) $$\int_0^T \sum_k H_k\, |\Delta\psi_k|^2 \le \frac{|v|^2}{A^2 H_1 \lambda_1^2}\, T + |\theta(0)|_{-1}^2$$

(3.β) $$|\theta(t)|_{-1}^2 \le |\theta(0)|_{-1}^2 \exp(-A\, c_1 t) + \frac{|v|^2}{A^2 H_1 \lambda_1^2 c_1} \,(1 - \exp(-A\, c_1 t))$$

and for $T > 0$

(3.γ) $$T\|\theta(T)\|_2^2 \le C(T, \Omega, A, H_k, p_k, |\theta(0)|_{-1}^2)$$

The mapping $\theta_k^0 \to \theta_k(.,t)$ is continuous in $H^{-1}(\Omega)$.

3.2 Proof of theorem.

As in (1.2), we implement a Galerkin method, and start with a priori estimates on θ_k. We recall that

(3.22) $$(J(f,g),f) = 0 \text{ for } f, g \text{ constant on } \partial\Omega$$

and integrating by parts, using Holder inequality and Sobolev imbeddings

(3.23) $$|(J(f,\Delta h),h)| = |(J(\Delta h,h),f)| \le c\, \|f\|_H{}^2\, |\nabla h^2|$$

As in part 1 and 2, we introduce $\xi = \Delta\psi$.

(i) We multiply (3.10_k) by $H_k\psi_k$, integrate over Ω and sum on k. Using (3.11) - (3.17) and the Green formula, we find :

(3.24) $$\frac{\partial}{\partial t}|\theta|_{-1}^2 + \sum_k A\, H_k\, |\Delta\psi_k|^2 + \varepsilon\, H_N\, |\nabla\psi_N|^2 \le \frac{|v|^2}{A\, H_1\, \lambda_1^2}\,.$$

It follows after integrating in time that

(3.25) $$\int_0^T \sum_k H_k\, |\Delta\psi_k|^2 \le \frac{|v|^2}{A^2 H_1 \lambda_1^2}\, T + |\theta(0)|_{-1}^2\,.$$

Notice that

$$|\theta|_{-1}^2 \le \sum_k H_k\, |\Delta\psi_k|^2/\lambda_1 + \sum_k 2p_k\, (|\psi_k|^2 + |\psi_{k+1}|^2)\,.$$

$$|\theta|_{-1}^2 \le \sum_k H_k\, |\Delta\psi_k|^2/\lambda_1 + 2\,p \sum_k |\psi_k|^2 \qquad \text{with } p = \sup\,(p_k + p_{k-1})\,.$$

$$|\theta|_{-1}^2 \le \sum_k H_k\, |\Delta\psi_k|^2/\lambda_1 + 2\,p \sum_k H_k|\Delta\psi_k|^2 / (\lambda_1^2\, H') \qquad \text{with } H' = \inf\,(H_k)\,.$$

So, there exists a constant c_1 such that

$$c_1\, |\theta|_{-1}^2 \le \sum_k H_k\, |\Delta\psi_k|^2$$

and (3.24) give us

$$\frac{\partial}{\partial t}|\theta|_{-1}^2 + A\,c_1|\theta|_{-1}^2 \le \frac{|v|^2}{A\,H_1\,\lambda_1^2}.$$

We can apply the Gronwall lemma to obtain

$$|\theta(t)|_{-1}^2 \le |\theta(0)|_{-1}^2 \exp(-A\,c_1\,t) + \frac{|v|^2}{A^2 H_1 \lambda_1^2 c_1}\,(1-\exp(-A\,c_1 t))\,. \tag{3.26}$$

(ii) We multiply (3.10_k) by $H_k\Delta\psi_k$, integrate over Ω and sum on k. Using (3.11), (3.17) and the Green formula, we find :

$$\sum_k \frac{1}{2}\left(H_k \frac{\partial}{\partial t}|\Delta\psi_k|^2 + p_k\frac{\partial}{\partial t}|\nabla\psi_{k+1}-\nabla\psi_k|^2\right) + \sum_k H_k\,A\,|\nabla\xi_k|^2 + \varepsilon\,H_N\,|\xi_N|^2 \le$$

$$\le \sum_k H_k\beta_0\,|\nabla\psi_k|\,|\xi_k| + |v|\,|\xi_1| + \sum_k p_k\,|\nabla\psi_k|\,|\xi_{k+1}|\,(|\nabla\xi_{k+1}-\nabla\xi_k|) +$$

$$+\sum_k p_k\frac{\partial}{\partial t}(C_{k+1}-C_k)\int_\Gamma \frac{\partial}{\partial\nu}(\psi_{k+1}-\psi_k)\,d\sigma$$

Let E denote the left-hand side of this inequality. Using (3.25), it follows

$$E \le \sum_k H_k\,\beta_0\,M_1\,|\xi_k| + |v|\,|\xi_1| + \sum_k M_1\frac{2p_k}{H_k}|\xi_{k+1}|\,(|\nabla\xi_{k+1}-\nabla\xi_k|) +$$

$$+\sum_k p_k\frac{\partial}{\partial t}(C_{k+1}-C_k)\int_\Omega(\Delta\psi_{k+1}-\Delta\psi_k)\,d\omega$$

and then there exist two constants c_3 and c_4 such that

$$E \le c_3 + c_4\sum_k|\xi_k|^2 + \sum_k\frac{A\,H_k}{2}|\nabla\xi_k|^2 + \sum_k p_k\left(\left|\frac{\partial C_{k+1}}{\partial t}\right| + \left|\frac{\partial C_k}{\partial t}\right|\right)(|\Delta\psi_{k+1}| + |\Delta\psi_k|)$$

hence , with the notation $p_0 = 0$,

$$E \le c_3 + c_4\sum_k|\xi_k|^2 + \sum_k (p_k + p_{k-1})\left|\frac{\partial C_k}{\partial t}\right|\left(\sum_j |\Delta\psi_j|\right) \tag{3.27}$$

To majorize the derivative in time of C(t), we need the following lemma (Bernier [3]).

<u>Lemma 1.</u> The solution $\psi = \psi' + C$, $\psi'(.,t) \in H_0^1$, $C(t) \in \mathbb{R}^N$ of the equations (3.10)-(3.17) verifies

$$\left|\frac{\partial C_k}{\partial t}\right| \le a_k\sum_j|\Delta\psi_j| + \alpha_k$$

where a_k, $\alpha_k \in \mathbb{R}$ depend on the data.

The proof of this lemma is given in Appendix C.

Let $p = \sup (p_k + p_{k-1})$, $a = \sup (a_k)$, the suprema are taken on $k = 1, N$.

$$\sum_k (p_k + p_{k-1}) \left|\frac{\partial C_k}{\partial t}\right| \left(\sum_j |\Delta \psi_j|\right) \leq p\, a\, N \sum_{i,j} |\Delta \psi_j|\, |\Delta \psi_i| + \sum_k \alpha_k\, (t) \sum_i |\Delta \psi_i|$$

$$\sum_k (p_k + p_{k-1}) \left|\frac{\partial C_k}{\partial t}\right| \left(\sum_j |\Delta \psi_j|\right) \leq c \sum_j |\Delta \psi_j|^2 + c'\, \alpha_j\, (t)^2$$

and finally (3.27) gives

$$\frac{1}{2}\frac{\partial}{\partial t} \|\theta\|_2^2 + \sum_k \frac{A H_k}{2} |\nabla \xi_k|^2 + \varepsilon H_N |\xi_k|^2 \leq c_5 + c_6 \sum_k |\xi_k|^2 \,. \tag{3.28}$$

We now multiply this inequality by t and integrate by parts, we have

$$\frac{\partial}{\partial t}(t\, \|\theta\|_2^2) + \sum_k A H_k t\ |\nabla \psi_k|^2 \leq c_5\, t + c_6\, t \sum_k |\xi_k|^2 + \|\theta\|_2^2 \,.$$

Integrating in time between 0 and T and using (3.25), we obtain

$$T\, \|\theta(t)\|_2^2 \leq C\left(T, \Omega,\ A,\ H_k,\ p_k,\ |\theta(0)|_{-1}^2\right) \text{for } T > 0. \tag{3.29}$$

Integrating now (3.28) between $\alpha > 0$ and T, and using (3.29), it follows

$$\int_\alpha^T \sum_k H_k |\nabla \xi_k|^2 \leq C'\left(T, \Omega,\ A,\ H_k,\ p_k,\ |\theta(0)|_{-1}^2\right). \tag{3.30}$$

We implement a Galerkin method using orthogonal basis $\{w_j\}_j$ of $H_0^1(\Omega)$ defined by $-\Delta w_j = \lambda_j w_j$. We denote by V_m the space spanned by $\{w_1, \ldots, w_m\}$. For each integer m, we look for an approximate solution $\{\theta_k^m\}$ of the form

$$\psi_k^m = \psi'^m_k + C_k^m,\ \ \theta_k^m = \theta'^m_k + (WC^m)_k \ \text{ with } \ \theta'^m_k(.,t),\ \psi'^m_k(.,t) \ \text{ in } H_0^1(\Omega)$$

$$\psi'^m_k(.,t) = \sum_k g^m_{kj}(t)\, w_j \ \text{ and } \theta'^m_k(.,t) = \sum_k l^m_{kj}(t)\, w_j$$

satisfying in $\Omega \times]\,0, T\,[$

$$\left(\frac{\partial \theta'^m_k}{\partial t}, w_j\right) + \left(J(\psi_k^m, \theta_k^m + f),\ w_j\right) - A\,(\Delta \xi_k^m, w_j) = \delta_{k1}\,(S_1, w_j) +$$

$$+ \delta_{k,N}\, \varepsilon\, H_N\, (\xi_N^m , w_j) - \frac{p_k}{H_k} \frac{\partial}{\partial t} \left(C_{k+1}^m - C_k^m\right) + \frac{p_{k-1}}{H_k} \left(\frac{\partial}{\partial t} (C_k^m - C_{k-1}^m)\right)$$

(where $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,j} = 1$ if $i = j$).

$$(\nabla\psi_k^m, \nabla w_k) = - (\xi_k^m, w_k)$$

$$\theta_k^m (.,0) = \theta_k^{m,0}(.) \qquad \text{in } \Omega$$

and $\quad \theta_k^{m,0} \to \theta_k^0$ in $H^{-1}(\Omega)$ when $m \to +\infty$, $\quad |\theta^{m,0}|_{-1} \leq |\theta^0|_{-1}$.

This system has a solution in any interval [0,T] (Bernier [3]).

Obviously, the solution verifies (3.25), (3.26), (3.29), (3.30) and so θ'^m is bounded independently of m in $L^2(0,T, L^2) \cap L^2(\alpha,T, H_0^1)$ for $\alpha > 0$. As in part 1, we can prove that $\frac{\partial\theta'^m}{\partial t}$ is bounded in $L^2(0,T, H^{-2})$ independently of m, and the lemma 1 of Appendix C gives that $\frac{\partial C_k^m}{\partial t}$ is bounded in $L^2(0,T)$ independently of m. So, we can extract subsequences such that

$\theta'^m \to \theta'$ in $L^2(0,T, L^2) \cap L^2(\alpha,T, H_0^1(\Omega))$ weakly.

$\frac{\partial\theta'^m}{\partial t} \to \frac{\partial\theta'}{\partial t}$ in $L^2(0,T, H^{-2})$ weakly.

$\frac{\partial C_k^m}{\partial t} \to \frac{\partial C_k}{\partial t}$ in $L^2(0,T)$ weakly.

(3.25), (3.26), (3.29) give (3.α), (3.β) and (3.γ).

The same arguments as in part 1 allow us to conclude that

$$\theta' \in \mathscr{C}([0,T], H^{-1}) \cap L^2(0,T, L^2) \cap L^2_{loc}(]0,T], H_0^1), \quad C \in \mathscr{C}([0,T]),$$

so $\quad \theta \in \mathscr{C}([0,T], H^{-1}) \cap L^2(0,T, L^2) \cap L^2_{loc}(]0,T], H^1)$

and that

$\langle J(\theta_k^m, \psi_k^m), w\rangle \to \langle J(\theta_k, \psi_k), w\rangle \qquad$ for w in $\mathscr{C}([0,T], W_0^{1,4}(\Omega))$.

<u>Proof of uniqueness and continuous dependance on the initial data.</u>

Let (ψ,θ), (ψ^*,θ^*) denote two solutions of the problem (1.10) - (1.17), corresponding to the initial values θ_0 and θ_0^*. We set $\Psi_k = \psi_k - \psi_k^*$, $\Xi_k = \theta_k - \theta_k^*$.

(i) (Ψ_k, Ξ_k) satisfies

$$(3.31_k) \qquad \frac{\partial \Xi_k}{\partial t} - A\,\Delta^2\Psi_k = \delta_{k,N}\,\varepsilon\,H_N\,\Delta\Psi_k - J(\psi_k, \theta_k + f) - J(\psi_k^*, \theta_k^* + f)$$

$$\Delta\Psi - W\Psi = \Xi$$

$$(3.32_k) \qquad \Xi_k(.,0) = \theta_k(.,0) - \theta_k^*(.,0)$$

$$(3.33_k) \qquad \Delta\Psi_k = 0 \qquad \text{on } \partial\Omega.$$

We notice that $J(\psi,\theta) - J(\psi^*,\theta^*) = J(\Psi,\theta) + J(\psi^*,\Xi)$.

Multiplying (3.31_k) by $H_k\Psi_k$, integrating over Ω and summing on k, we find

$$E = \frac{1}{2}\frac{\partial}{\partial t}|\Xi|_{-1}^2 + \sum_k A\,H_k\,|\Delta\Psi_k|^2 + \varepsilon H_N\,|\nabla\Psi_N|^2 \leq$$

$$\leq \sum_k 2\,c\,|\nabla\Psi_k|_{L^4}^2\,|\Delta\Psi_k^*| + \sum_k p_k \frac{2C'}{H_k}|\Psi_k - \Psi_{k+1}|\,|\Delta\Psi_k| + \sum_k |\nabla\Psi_k|^2$$

thanks to the Green formula, the Holder inequality and (3.23), where C' is the constant of (3.30) and we denote by c any constant depending on Ω.

$$E \leq c\sum_k |\theta_k^*|\,|\nabla\Psi_k|_{L^2}^{\sigma}\,|\Delta\Psi_k|_{L^2}^{2-\sigma} + \frac{A}{2}\sum_k |\Delta\Psi_k|^2 + c\,|\Xi|_{-1}^2\,.$$

Thanks to the Young inequality, we obtain

$$E \leq \frac{A}{2}\sum_k |\Delta\Psi_k|^2 + c\sum_k |\theta_k^*|\,|\nabla\Psi_k|^2 + \frac{A}{2}\sum_k |\Delta\Psi_k|^2 + c\,|\Xi|_{-1}^2\,.$$

Since $|\theta_k^*| \leq C$ by the a priori estimate (3.29), there exists a constant K depending on T, Ω, A, H_k, p_k, $|\theta(0)|_{-1}$ such that

$$\frac{\partial}{\partial t}|\Xi|_{-1}^2 \leq K\,|\Xi|_{-1}^2\,.$$

Integrating in time between 0 and T, we conclude with

$$|\Xi(t)|_{-1}^2 \leq |\Xi(0)|_{-1}^2 \exp(K\,t).$$

PART 4. THE MAXIMAL ATTRACTOR.

4.1 Existence of absorbing sets and of a maximal attractor.

Theorem 4.1. The semi-group $S(t)$ from H^{-1} to H^{-1}, $S(t)\theta_0(.) = \theta(.,t)$, $\theta = (\theta_1,..., \theta_N)$, associated to (3.10) - (3.17) is such that

(i) There exist bounded absorbing sets in H^{-1} and L^2.

(ii) There exists a maximal attractor $\mathcal{A}$ which is bounded in L^2, compact and connected in H^{-1}. Its basin of attraction is the whole space H^{-1}.

Proof of theorem 4.1.

(i) Absorbing sets in H^{-1}.

In part 3, we have obtained an inequality that we rewrite

$$(3.\beta) \qquad |\theta(t)|_{-1}^2 \le |\theta(0)|_{-1}^2 \exp(-A c_1 t) + \frac{|v|^2}{A^2 H_1 \lambda_1^2 c_1}(1 - \exp(-A c_1 t))$$

We deduce that any ball of H^{-1} centered at 0 and of radius $r_2^2 > r_1^2 = \dfrac{|v|^2}{A^2 H_1 \lambda_1^2 c_1}$ is an absorbing set in H^{-1}. Indeed, if $\mathcal{B}$ is a bounded set of H^{-1} included in a ball $B(0,R)$ of H^{-1} centered at 0 of radius R, then $S(t)\mathcal{B} \subset B(0,r_2)$ for $t \ge T_1 = T_1(\mathcal{B}, r_2)$

$$T_1 = \frac{1}{A c_1} \log\left(\frac{R^2}{r_2^2 - r_1^2}\right)$$

Absorbing set in L^2.

We first integrate (3.24)

$$\frac{\partial}{\partial t}|\theta|_{-1}^2 + \sum_k A H_k |\Delta\psi_k|^2 + \varepsilon H_N |\nabla\psi_N|^2 \le \frac{|v|^2}{A H_1 \lambda_1^2}$$

between t and t + r, r > 0 fixed. We obtain

$$|\theta(t+r)|_{-1}^2 + \sum_k A\ H_k \int_t^{t+r} |\Delta\psi_k|^2 + \varepsilon H_N \int_t^{t+r} |\nabla\psi_N|^2 \le \frac{|v|^2}{A H_1 \lambda_1^2} r + |\theta(t)|_{-1}^2 .$$

Then, for $t \ge T_1$

$$\sum_k A\ H_k \int_t^{t+r} |\Delta\psi_k|^2 \le r \frac{|v|^2}{A\ H_1 \lambda_1^2} + r_2 \text{ and since } \sum_k H_k |\nabla\psi_k|^2 \le |\theta(t)|_{-1}^2 \le r_2^2 ,$$

and

$$\int_t^{t+r} \|\theta\|_2^2 \le \sum_k H_k \int_t^{t+r} |\Delta\psi_k|^2 + \frac{p}{H'} \sum_k H_k \int_t^{t+r} |\nabla\psi_k|^2$$

we have

$$\int_t^{t+r} \|\theta\|_2^2 \le \frac{|v|^2\, r}{A^2 H_1 \lambda_1{}^2} + \frac{p}{H'} r_2^2 r + \frac{r_2}{A} . \tag{4.2}$$

Let us rewrite the inequality (3.28) we obtain in part 3.

$$\frac{1}{2}\frac{\partial}{\partial t} \|\theta\|_2^2 + \sum_k \frac{A H_k}{2} |\nabla\xi_k|^2 + \varepsilon H_N |\xi_N|^2 \le c_5 + c_6 \sum_k |\xi_k|^2$$

so

$$\frac{\partial}{\partial t} \|\theta\|_2^2 \le c_7 + c_8 \|\theta\|_2^2 .$$

We apply the uniform Gronwall lemma with

$$y = \|\theta\|_2^2 \ , g = c_8 \ , h = c_7$$

and use (4.2). We conclude

$$y(t+r) \le (c_9 + c_7\, r) \exp\, (c_8 r)$$

where $c_9 = \dfrac{|v|^2}{A^2 H_1 \lambda_1{}^2} + \dfrac{p}{H'} r_2^2 + \dfrac{r_2}{Ar}$

that we rewrite

$$\|\theta\|_2^2 \le r_3^2 \ \text{ for } t > T_1 + r. \tag{4.3}$$

The ball $\mathcal{B}_1$ of L^2 centered at 0 of radius r_3 is absorbing in L^2. As in part 2, this result provides the uniform compactness of S(t) in H^{-1}. If $\theta_0 \in \mathcal{B} \subset B(0,R)$ and $t \ge T_1 + r$, $\theta(.,t)$ belongs to $\mathcal{B}_1$ which is bounded in L^2 and relatively compact in H^{-1}.

To conclude, the assumptions (2.1) and (2.2) are satisfied and the theorem 2.4 gives us theorem 4.1. (ii).

4.2 Dimension of the maximal attractor.

It is easy to prove that we have the same lemma 2.9 as in part 2, written for a system of equations. Let us introduce

$$F(\theta)_k = A\ \Delta^2\psi_k + J(\theta_k + f, \psi_k) + S_k + F_k. \tag{4.4}$$

We rewrite

(4.5) $$\frac{\partial\theta}{\partial t} = F(\theta).$$

As in part 3, we now intend to estimate $Tr(F'(\theta(\tau))oQ_m(\tau))$.
We introduce

(4.6) $$\frac{\partial}{\partial t}(U) = F'(\theta)\,U \qquad \text{in } \Omega \times \mathbb{R}^+$$

where U is the vector of composant U_k such that :

(4.7_k) $$(F'(\theta)\,U)_k = A\,\Delta^2 V_k + J(U_k, \psi_k) + J(\theta_k + f, V_k) - \delta_{N,k}\,\varepsilon\,\Delta V_N$$

(4.8) $$\Delta V - W\,V = U \qquad \text{in } \Omega \times \mathbb{R}^+$$

(4.9_k) $$U_k(.,0) = U_k^0(.) \qquad \text{in } \Omega$$

and (ψ,θ) the solution of (4.5).

We consider $\{\phi^j(\tau)\}_j$ an orthonormal basis of $Q_m(\tau)H^{-1} = \text{Vect}\,[U^1, ..., U^m]$ for the scalar product $<<\,,\,>>$ and introduce $\varphi^j \in H^1_c$ defined by

(4.10) $$\Delta\varphi^j - W\,\varphi^j = \phi^j \qquad \varphi^j = \varphi'^j + C^j \qquad \varphi'^j \in H^1_0.$$

So

$$<< F'(\theta)\phi^j, \phi^j >> = \sum_k \Big(-A < \Delta^2\varphi^j_k, (H\varphi'^j)_k > - < J(\phi^j_k,\psi_k), (H\varphi'^j)_k >$$
$$- < J(\theta_k, \varphi^j_k), (H\varphi'^j)_k > + < \delta_{N,k}\,\varepsilon\,\Delta\varphi^j_N\,,\ (H\varphi'^j)_k > \Big).$$

$$<< F'(\theta)\phi^j, \phi^j >> = \sum_k \Big(-A\,H_k\,|\Delta\varphi^j_k|^2 - < J(\phi^j_k, \psi_k), (H\varphi'^j)_k> - \delta_{N,k}\,\varepsilon H_k\,|\nabla\varphi^j_k|^2\Big)$$

(4.11) $$<< F'(\theta)\phi^j, \phi^j >> = \sum_k \Big(-A\,H_k\,|\Delta\varphi^j_k|^2 - \delta_{N,k}\,\varepsilon H_k\,|\nabla\varphi^j_k|^2 +$$
$$+ < J(\varphi^j_k, \psi_k), H_k\,\Delta\varphi^j_k > - < J(\varphi^j_k, \psi_k), H_k\,(W\varphi^j)_k > \Big).$$

As in part 2, we have the (2.23) inequality

$$\Big|\sum_j H_k < J(\varphi^j_k,\psi_k), \Delta\varphi^j_k > \Big| \le 2\,\|\psi'_k\|_H^2\,|\rho_k|$$

where $\rho_k = H_k \sum_k \left(\frac{\partial\varphi^j_k}{\partial x}\right)^2 + \left(\frac{\partial\varphi^j_k}{\partial y}\right)^2$. We introduce $\rho = \sum_k \rho_k$ and then

(4.12) $$\Big|\sum_k \sum_j H_k < J(\varphi^j_k,\psi_k), \Delta\varphi^j_k > \Big| \le 2\,N\,\|\psi'\|_H^2\,|\rho|$$

We now look for an estimate of $\left|\sum_k \sum_j H_k < J(\varphi_k^j,\psi_k), (W\varphi^j)_k > \right|$

Let us consider the sum on j, j = 1, ..., m :

$$\left|\sum_j H_k < J(\varphi_k^j,\psi_k), (W\varphi^j)_k > \right| = \left|\sum_j p_k < J(\varphi_k^j,\psi_k), \varphi_{k+1}^j > \right|$$

$$+\left|\sum_j p_{k-1} < J(\varphi_k^j,\psi_k), \varphi_{k-1}^j > \right|$$

As

$$\left|\sum_j p_{k-1} < J(\varphi_k^j,\psi_k), \varphi_{k-1}^j > \right| = \left|\sum_j p_{k-1} < J(\varphi_{k-1}^j, \varphi_k^j), \psi_k > \right|$$

$$\left|\sum_j p_{k-1} < J(\varphi_k^j,\psi_k), \varphi_{k-1}^j > \right| \leq \int |\psi_k| \frac{1}{2} \left(\frac{1}{H_k} \rho_k + \frac{1}{H_{k-1}} \rho_{k-1} \right) p_{k-1}$$

then

$$\left|\sum_k \sum_j H_k < J(\varphi_k^j,\psi_k), (W\varphi^j)_k > \right| \leq \frac{p}{2 \inf (H_k)} \|\psi'\|_H^2 \, |\rho| . \tag{4.13}$$

We recall that $p = \sup (p_k + p_{k-1})$. Combining (4.11), (4.12), (4.13), we obtain

$$\sum_j << F'(\theta)\phi^j, \phi^j >> = \sum_j \sum_k -A\ H_k |\Delta\varphi_k^j|^2 - \delta_{N,k}\, \varepsilon\, H_k \sum_j |\nabla \varphi_k^j|^2 \tag{4.14}$$

$$+ \left(2\,N + \frac{p}{2 \inf (H_k)}\right) \|\psi'\|_H^2 \, |\rho| .$$

Now, let us introduce two families of vectors $\{u_j\}$and $\{v_j\}$ defined by

$$u_j = {}^t\left(\sqrt{H_1}\, \frac{\partial\varphi_1^j}{\partial x}, \sqrt{H_1}\, \frac{\partial\varphi_1^j}{\partial y}, \ldots, \sqrt{H_N}\, \frac{\partial\varphi_N^j}{\partial x}, \sqrt{H_N}\, \frac{\partial\varphi_N^j}{\partial y}\right)$$

$$v_j = {}^t\left(\sqrt{p_1}\, (\varphi_1^j - \varphi_2^j), \ldots, \sqrt{p_{N-1}}\, (\varphi_{N-1}^j - \varphi_N^j)\right)$$

and $w_j = {}^t (u_j, v_j)$.

We notice that $\rho(x) = \sum_j (u_j(x))^2$ and introduce $\sigma(x) = \sum_j (v_j(x))^2$.

As $\{\phi_j\}$ is an orthonormal family for <<.,.>>, $\{w_j\}$ is an orthonormal family for $(L^2(\Omega))^{3N-1}$ and the $\{u_j\}$ family is suborthonormal, i.e.

$$\sum_i \sum_k \zeta_i\zeta_k \int u_i u_k \leq \sum_i \zeta_i^2 \text{ for all } \zeta_i, \zeta_k \text{ in } \mathbf{R} \rightarrow$$

We can apply the generalization of the Sobolev-Lieb-Thirring inequality (Ghidaglia-Marion-Temam [8]) :

$$\int \rho(x)^2\,dx \le K_1 \sum_j \int (\nabla u_j)^2\,dx + K_2 \int \rho(x)\,dx$$

where K_1 K_2 depend only on Ω. As $\{w_j\}$ is orthonormal, $\int \rho(x)\,dx \le m$, and

(4.15) $$\int \rho(x)^2\,dx \le K_3 \sum_j \sum_k H_k\,|\Delta\varphi_k^j|^2 + K_4\,m\,.$$

We now replace in (4.14)

(4.16) $$\sum_j \langle\langle F'(\theta)\phi^j, \phi^j\rangle\rangle \le \sum_j \sum_k -A\,H_k|\Delta\varphi_k^j|^2 - \delta_{N,k}\,\varepsilon\,H_k \sum_j |\nabla\varphi_N^j|^2$$

$$+ K_6\,\|\psi'\|_{H^2}^2 + \frac{A}{2} \sum_j \sum_k H_k\,|\Delta\varphi_k^j|^2 + A\frac{K_4}{2\,K_3}m \text{ for all } \alpha > 0.$$

where $K_6 = \frac{K_3}{2A}\,(2N + \frac{p}{2H'})^2$.

(4.17) $$\sum_j \langle\langle F'(\theta)\phi^j, \phi^j\rangle\rangle \le -\frac{A}{2}\sum_j \sum_k H_k\,|\Delta\varphi_k^j|^2 + K_6\,\|\psi'\|_{H^2}^2 + A\frac{K_4}{2\,K_3}m$$

Let us now bound from below the Laplacian term. Since $\int (\rho(x) + \sigma(x))\,dx = m$, we have

$$m^2 \le 2\,|\Omega| \left(\int \rho^2(x)\,dx + \int \sigma^2\,(x)\,dx \right)$$

The Sobolev-Lieb-Thirring inequality (Temam[14]) gives

$$m^2 \le 2\,|\Omega|\,æ \left(\sum_j \sum_k c_y H_k\,|\Delta\varphi_k^j\,(x)|^2 + p_k|\nabla\varphi_k^j\,(x) - \nabla\varphi_{k+1}^j\,|^2 \right)$$

with æ depending only on Ω.

$$m^2 \le 2\,|\Omega|\,æ\,(c_y + 2p/(\lambda_1 H')) \left(\sum_j \sum_j H_k\,|\Delta\varphi_k^j\,(x)\,|^2 \right)$$

and then

$$\sum_j \sum_k H_k\,|\Delta\varphi_k^j\,(x)\,|^2 dx \ge m^2 / 2\,|\Omega|\,æ\,(c_y + 2p/(\lambda_1 H'))$$

We now report in (4.17) and obtain

(4.18) $$\sum_j \langle\langle F'(\theta)\phi^j, \phi^j\rangle\rangle \le -\,m^2 \frac{A}{4\,|\Omega|\,æ\,(c_y + 2p/(\lambda_1 H'))} + $$

$$+ m\,A\frac{K_4}{2K_3} + K_6\,\|\psi'\|_{H^2}^2$$

Let us now rewrite

(3.24) $$\frac{\partial}{\partial t}|\theta|_{-1}^2 + \sum_k A\,H_k\,|\Delta\psi_k|^2 + \varepsilon H_N\,|\nabla\psi_N|^2 \le \frac{|v|^2}{AH_1\lambda_1^2}$$

After integrating in time between 0 and t, we obtain

$$|\theta(t)|_{-1}^2 + \sum_k A\ H_k \int_0^t |\Delta\psi_k|^2 + \varepsilon H_N \int_0^t |\nabla\psi_N|^2 \leq \frac{|v|^2}{AH_1\lambda_1^2}\ t + |\theta(0)|_{-1}^2$$

and since $c_y \sum_k |\Delta\psi_k(s)|^2 \geq \|\psi'(s)\|_{H^2}^2$

$$\limsup_{t \to +\infty} \left(\frac{1}{t} \int_0^t \|\psi'(s)\|_{H^2}^2\, ds \right) \leq \frac{|v|^2 c_y}{A^2 H' H_1 \lambda_1^2} = K_7$$

Combining with (4.18),

$$q_m \leq -\, m^2 \frac{A}{4\,|\Omega|\,æ\,(c_y + 2p/(\lambda_1 H'))} + m\,A \frac{K_4}{2K_3} + K_6 K_7\,.$$

Let m' be an integer such that $q_{m'} < 0$. The Hausdorff dimension is majorized by m'.

The expression of the bound of m' is :

$$m' \leq 1 + 2\,|\Omega|\,æ\,\left(c_y + 2p/(\lambda_1 H')\right) \frac{K_4}{K_3} +$$

$$+ \left(2\,|\Omega|\,æ\,(c_y + 2p/(\lambda_1 H'))\, c_y \frac{K_3}{H'\,H_1\,\lambda_1^2}\right)^{1/2} \left(2N + \frac{p}{2H'}\right) \frac{|v|}{A^2}$$

with æ, K_3, K_4, λ_1, c_y depending only on Ω, $p = \sup(p_k + p_{k-1})$, $H' = \inf(H_k)$ and N the number of layers.

$$m' \leq C_1 + C_2 \frac{|v|}{A^2}\,.$$

For the fractal dimension, we introduce $f(m) = -\,a_1 m^2 + a_2 m + a_3$ with

$$a_1 = \frac{A}{4\,|\Omega|\,æ\,(c_y + 2p/(\lambda_1 H'))} \qquad a_2 = A\frac{K_4}{2K_3} \qquad a_3 = K_6 K_7$$

and $S = \sup(f(m)) = a_2^2 / (2a_1) + a_3$. We choose m" such that $q_{m''} < -\,\alpha S$, $\alpha > 0$, then

$$\dim_F(\mathcal{A}) \leq m''\,(1 + 1/\alpha).$$

APPENDIX A.

We here prove the lemma 2.9.

1) S(t) is uniformly differentiable on $\mathcal{A}$. We note by

$$w(.,t) = v(.,t) - u(.,t) = S(t)v_0 - S(t)u_0$$
$$U(t) = L(t,u)\,(v_0 - u_0).$$

Our aim is to prove that

$$\sup_{\substack{u_0, v_0 \in \mathcal{A} \\ 0 < \|v_0 - u_0\|_{H^{-1}} < \varepsilon}} \frac{\|w(t) - U(t)\|_{H^{-1}}}{\|v_0 - u_0\|_{H^{-1}}} \to 0 \text{ as } \varepsilon \to 0$$

w(.,t) verifies the equations

(A.1) $\quad \dfrac{\partial w}{\partial t} = -\varepsilon\, w - \dfrac{\partial \Phi}{\partial x} + A\, \Delta w - J(\Phi,v) - J(\Delta^{-1} u,w)$

(A.2) $\quad \Delta\Phi = w$

(A.3) $\quad w(t = 0) = v_0 - u_0$

(A.4) $\quad w = 0$ and $\Phi = 0$ on Γ.

U(.,t) verifies the equations (2.16) - (2.19) with $\xi = u$ and $U_0 = v_0 - u_0$.

(A.5) $\quad \dfrac{\partial U}{\partial t} = -\varepsilon\, U - \dfrac{\partial V}{\partial x} + A\, \Delta U - J(\Delta^{-1} u,U) - J(V,u)$ in $\Omega \times \mathbb{R}^+$

(A.6) $\quad \Delta V = U$ in $\Omega \times \mathbb{R}^+$

(A.7) $\quad U = 0$ and $V = 0$ on Γ.

(A.8) $\quad U(t = 0) = v_0 - u_0 = w\,(t = 0)$

Let us introduce $z(.,t) = \Phi(.,t) - V(.,t)$ and $y(.,t) = w(.,t) - U(.,t)$.

Then y and z are solutions of the equations :

(A.9) $\quad \dfrac{\partial y}{\partial t} = -\varepsilon\, y - \dfrac{\partial z}{\partial x} + A\, \Delta y - J(\Delta^{-1} u,y) - J(z,u) - J(\Phi,w)$

(A.10) $\quad \Delta z = y$

(A.11) $\quad y = 0$ and $z = 0$ on Γ.

(A.12) $\quad z(t = 0) = 0$

We multiply the equation (A.9) by z and integrate on Ω, using the same tools as in part 1, we obtain :

(A.13) $\quad \dfrac{1}{2}\dfrac{\partial}{\partial t} |\nabla z|^2 + \varepsilon\, |\nabla z|^2 + A\, |\Delta z|^2 \le |\langle J(\Delta^{-1} u,\Delta z),z\rangle| + |\langle J(\Phi,\Delta\Phi), z\rangle|$

We have proved in (2.21) that $|\langle J(f,\Delta f),g\rangle| \le c\, \|g\|_{H^2}\, |\nabla f|^2_{L^4}$, so

$$|<J(\Delta^{-1} u,\Delta z),z>| \leq c\, |\nabla z|^2_{L^4}\ \|\Delta^{-1} u\|_H{}^2 .$$

We use the Holder inequality and we denote by c any constant depending on Ω.

$$|<J(\Delta^{-1} u,\Delta z),z>| \leq c\, |u|\, |\nabla z|^{\sigma}_{L^2}\, |\Delta z|^{2-\sigma}_{L^2} .$$

Thanks to the Young inequality, we obtain

$$\text{(A.14)} \qquad |<J(\Delta^{-1} u,\Delta z),z>| \leq c\, |u|\, |\nabla z|^2_{L^2} + \frac{A}{2} |\Delta z|^2_{L^2} .$$

Since v_0, $u_0 \in \mathcal{C}$, then $v,u \in \mathcal{C}$ and, using (2.9), $|u| < r_3$. So,

$$\text{(A.15)} \qquad |<J(\Delta^{-1} u,\Delta z),z>| \leq c\, |\nabla z|^2_{L^2} + \frac{A}{2} |\Delta z|^2_{L^2} .$$

Let us now bound from above $|<J(\Phi,\Delta\Phi), z>|$. We recall that $|\,.\,| = |\,.\,|_{L^4}$.

$$|<J(\Phi,\Delta\Phi), z>| = |<J(z,\Phi'),\Delta\Phi>| \leq |\nabla z|_{L^4}\, |\Delta\Phi|\, |\nabla\Phi|_{L^4}.$$

Since $\Delta\Phi = w$, using Sobolev embeddings, we obtain

$$|<J(\Phi,\Delta\Phi), z>| \leq c\, |\Delta z|\, |w|\, (|\Delta\Phi|^{1-\sigma}_{L^2}\ |\nabla\Phi|^{\sigma}_{L^2}) \qquad \text{with } 0 \leq \sigma \leq 1/4.$$

$$|<J(\Phi,\Delta\Phi), z>| \leq c\, |\Delta z|\, (|w|^{2-\sigma}_{L^2}\ |\nabla\Phi|^{\sigma}_{L^2}) \qquad \text{with } 0 \leq \sigma \leq 1/4.$$

We set $\sigma' = 1 - \sigma$. So

$$|<J(\Phi,\Delta\Phi), z>| \leq c\, |\Delta z|\, (|w|^{1+\sigma'}\, |\nabla\Phi|^{1-\sigma'}).$$

$$|<J(\Phi,\Delta\Phi), z>| \leq \frac{A}{2} |\Delta z|^2 + c\, |w|^2\, |w|^{2\sigma'}\, |\nabla\Phi|^{2-2\sigma'}.$$

Since $v,u \in \mathcal{C}$, $|u| < r_3$, , $|v| < r_3$ and then , $|w| = |u-v| < 2r_3$. The last inequality gives

$$\text{(A.16)} \qquad |<J(\Phi,\Delta\Phi), z>| \leq \frac{A}{2} |\Delta z|^2 + c\, |w|^2\, (2r_3)^{2\sigma'}\, |\nabla\Phi|^{2-2\sigma'}.$$

Replacing (A.15) and (A.16) in (A.13), it results

$$\frac{1}{2}\frac{\partial}{\partial t} |\nabla z|^2 + \varepsilon\, |\nabla z|^2 + A\, |\Delta z|^2 \leq c\, |\nabla z|^2 + A\, |\Delta z|^2 + c\, |w|^2\, |\nabla\Phi|^{2-2\sigma'}.$$

So

$$\text{(A.17)} \qquad \frac{\partial}{\partial t} |\nabla z|^2 \leq c\, |\nabla z|^2 + c\, |w|^2\, |\nabla\Phi|^{2-2\sigma'}.$$

Multiplying (A.1) by Φ and integrating on Ω, we obtain a result similar to (1.18)

$$\frac{1}{2}\frac{\partial}{\partial t} |\nabla\Phi|^2 + \varepsilon\, |\nabla\Phi|^2 + A\, |w|^2 \leq |<J(\Delta^{-1} u,\Delta\Phi),\Phi>|.$$

Thanks to (A.14), we have

$$\frac{1}{2}\frac{\partial}{\partial t} |\nabla\Phi|^2 + \varepsilon\, |\nabla\Phi|^2 + A\, |w|^2 \leq c\, r_3\, |\nabla\Phi|^2_{L^2} + \frac{A}{2} |w|^2_{L^2}$$

that we rewrite

$$\frac{\partial}{\partial t} |\nabla\Phi|^2 + A\, |w|^2 \leq c\, |\nabla\Phi|^2 .$$

Integrating in time, it follows for $0 \leq t \leq T$

$$|\nabla\Phi(t)|^2 \leq |\nabla\Phi(0)|^2 \exp(\,c\,T\,) \qquad \text{and} \qquad \int_0^t |w(s)|^2\, ds \leq |\nabla\Phi(0)|^2 \exp(\,c\,T\,).$$

Then (A.17) becomes

$$\frac{\partial}{\partial t} |\nabla z|^2 \leq c\,|\nabla z|^2 + c\,|w|^2\,|\nabla\Phi(0)|^{2-2\sigma} \exp(\,c\,T\,(1-\sigma)\,).$$

Integrating (A.17) in time, we deduce that

$$|\nabla z(t)|^2 \leq c\,|\nabla\Phi(0)|^{2-2\sigma} \exp(\,c\,T\,(1-\sigma)\,)\,|\nabla\Phi(0)|^2 \exp(\,c\,T\,) \exp(\,c\,t\,)$$

then

$$\frac{|\nabla z(t)|^2}{|\nabla\Phi(0)|^2} \leq c\,|\nabla\Phi(0)|^{2-2\sigma} \exp(\,c'\,T\,) \to 0 \text{ as } |\nabla\Phi(0)| \to 0\,.$$

ii) Proof of (2.20)

$$\text{(2.20)} \qquad \sup_{\xi_0 \in \mathcal{A}} |L(t_0, \xi_0)|_{\mathcal{L}(H^{-1})} < +\infty\,.$$

We multiply (A.5) by V and integrate on Ω. We obtain thanks to (A.14)

$$\frac{1}{2}\frac{\partial}{\partial t} |\nabla V|^2 + \varepsilon\,|\nabla V|^2 + A\,|\Delta V|^2 \leq |\langle J(\Delta^{-1} u, \Delta z), z\rangle| \leq c\,|u|\,|\nabla V|^2 + \frac{A}{2}|\Delta V|^2\,.$$

We rewrite

$$\frac{\partial}{\partial t} |\nabla V|^2 \leq c\,r_3\,|\nabla V|^2$$

Integrating in time

$$|\nabla V(t)|^2 \leq |\nabla V(0)|^2 \exp\,(c\,r_3\,t)$$

The proof is completed.

APPENDIX B.

To prove that $\|\theta\|_2$ is a norm on L^2, we prove that $|\nabla\psi|$ is a norm on H^1_c, equivalent to the usual norm. We recall that

$$H^1_c = \{\psi' + C,\ \psi' \in H^1_0,\ C \in (\mathbb{R}^N)^2,\ C^M_1 = 0,\ \int_\Omega \psi_1\, d\omega = \int_\Omega \psi_k\, d\omega \text{ for } 1 \le k \le N\},$$

$\Lambda = \mathcal{B}^{-1}\, W\, \mathcal{B}$ and $\psi^M = \mathcal{B}^{-1}\,\psi$, $C^M = \mathcal{B}^{-1}C$, etc.

From (3.16) and (3.17), we deduce that $\psi^M \in H^1_M$ where

$$H^1_M = H^1_0 \times \left\{\psi'^M_k + C^M_k,\ \psi'^M_k \in H^1_0(\Omega),\ \int(\psi'^M_k + C^M_k)\, d\omega = 0\right\}^{N-1}$$

i.e.

$$H^1_M = H^1_0 \times \left\{\psi'^M_k - \frac{1}{\text{mes}(\Omega)} \int \psi'^M_k\, d\omega,\ \psi'^M_k \in H^1_0(\Omega)\right\}^{N-1}.$$

We have just to prove that the Poincare inequality is verified in H^1_c. Obviously,

$$|\psi^M_k|^2 = |\psi'^M_k|^2 - \frac{1}{\text{mes}(\Omega)}\left(\int \psi'^M_k\, d\omega\right)^2 \le |\psi'^M_k|^2$$

and so $\qquad |\psi^M| \le |\psi'^M|$

Since $\psi_k = \sum_j b_{k,j}\, \psi^M_j$, we have $\qquad |\psi| \le N \operatorname{Sup}_{k,j} (|b_{k,j}|)\, |\psi^M| = N\, b\, |\psi^M|$

$\psi'_k = \sum_j b^{-1}_{k,j}\, \psi'^M_j$, $\qquad |\psi'^M| \le N \operatorname{Sup}_{k,j} (|b^{-1}_{k,j}|)\, |\psi'| = N\, b^{-1}\, |\psi'|$.

where b and b^{-1} are the suprema for all k and j of $|b_{k,j}|$ and $|b^{-1}_{k,j}|$.

These three inequalities proved that

$$|\psi| \le N\, b\, |\psi^M| \le N\, b\, |\psi'^M| \le N^2\, b\, b^{-1}\, |\psi'|$$

Hence, since $\psi' \in H^1_0(\Omega)$,

$$|\psi| \le c_1\, |\psi'| \le c_2\, |\nabla\psi'| = c_2\, |\nabla\psi|.$$

APPENDIX C.

Proof of Lemma 1.

We rewrite the system of equations :

(C.1) $\frac{\partial}{\partial t}(\theta_k) = G_k$

with $G_k = J(\psi_k, \theta_k + f) + A\,\Delta^2\,\psi_k + S_k + F_k$

(C.2) $\Delta\psi - W\psi = \theta$

(C.3) $\theta(.,0) = \theta^0$

(C.4) $\Delta\psi = 0$ on $\partial\Omega$

(C.5) $\psi(.,t) = C(t)$ on $\partial\Omega$

(C.6) $\int_\Omega \psi(.,0)\, d\omega = 0$ where $\Delta\psi(.,0) = \Delta\psi^0(.) = \xi^0(.)$

(C.7) $\int_\Omega \psi_1\, d\omega = \int_\Omega \psi_k\, d\omega = \ldots = \int_\Omega \psi_N\, d\omega$ for $1 \le k \le N$,

(C.8) $\sum_k b^{-1}_{1k}\, C_k = 0$

and the Lemma 1 :

The solution $\psi = \psi' + C$, $\psi'(.,t) \in H^1_0$, $C(t) \in \mathbb{R}^N$ of the equations (C.1) - (C.8) verifies

$$\left|\frac{\partial C_k}{\partial t}\right| \le a_k \sum_{j=1}^{N} |\Delta\psi_j| + \alpha_k$$

where $a_k, \alpha_k \in \mathbb{R}$ depend on the data.

We first prove the following lemma

Lemma 2. The solution ψ of the equations (C.1) - (C.8) verifies

$$\int_\Omega G_k(t)\varphi_0(t)\, d\omega \le q_k\, |\Delta\psi_k(t)| + P_k,\ \forall\ \varphi_0 \in \mathscr{C}([0,T], H^2(\Omega) \cap W_0^{1,\infty}(\Omega))$$

where $q_k, P_k \in \mathbb{R}$ depend only on Ω, T, v, ξ_0 and φ_0.

Proof. From (C.1), we have

$$G_k = A\,\Delta\xi_k + S_k + F_k + J(\psi_k, \xi_k - (W\psi)_k + f).$$

We multiply the equation by φ_0 and integrate on Ω. Let us bound each term from above.

$$\int_\Omega A\,\Delta\xi_k.\varphi_0\, d\omega = \int_\Omega A\,\xi_k.\Delta\varphi_0\, d\omega \le A\ |\xi_k|\ |\Delta\varphi_0|\,.$$

$\int S_1.\varphi_0 \, d\omega \ \le |v| \, |\varphi_0| \, / \, H_1$.

$\int F_N.\varphi_0 \, d\omega = -\varepsilon \int \xi_N.\varphi_0 \, d\omega \le \varepsilon \, |\xi_N| \, |\varphi_0|$.

$$\int J(\psi_k, \xi_k).\varphi_0 \, d\omega = \int J(\varphi_0, \psi_k).\xi_k \, d\omega \le \|\nabla\varphi_0\|_\infty \, |\nabla\psi_k| \, |\xi_k|$$

$$\le \|\nabla\varphi_0\|_\infty \, M_1 \, |\xi_k| \qquad \text{thanks to (3.25).}$$

$$\int J(\psi_k, f).\varphi_0 \, d\omega = \beta \int \frac{\partial\psi_k}{\partial x}.\varphi_0 \, d\omega \le \beta \, |\varphi_0| \, |\nabla\psi_k| \ \le \ \beta \, M_1 \, |\varphi_0| \, .$$

$$\int J(\psi_k, \psi_{k+1}).\varphi_0 \, d\omega = \int J(\varphi_0, \psi_k). \, \psi'_{k+1} \, d\omega \le \|\nabla\varphi_0\|_\infty \, |\nabla\psi_k| \, |\psi'_{k+1}|$$
$$\le c \, \|\nabla\varphi_0\|_\infty \, |\Delta\psi_k| \, |\nabla\psi_{k+1}| \ \le c \, \|\nabla\varphi_0\|_\infty \, |\xi_k| \, M_1 \, .$$

It follows

$$\left|\int G_k.\varphi_0 \, d\omega\right| \ \le |\xi_k| \ \Big(A \, |\Delta\varphi_0| + \varepsilon|\varphi_0| + \|\nabla\varphi_0\|_\infty \, M_1 \, (1 + (p_k + p_{k-1})c/H_1)\Big) + \\ + \left(\frac{|v|}{H_1} + \beta M_1\right) |\varphi_0| \, .$$

As $\varphi_0 \in \mathscr{C}([0,T], H^2(\Omega) \cap W_0^{1,\infty}(\Omega))$, we are allowed to introduce

$$q_k = \sup_{t \in [0,T]} \ \Big(A \, |\Delta\varphi_0| + \varepsilon \, |\varphi_0| + \|\nabla\varphi_0\|_\infty \, M_1 \, (1 + (p_k + p_{k-1})c/H_1)\Big)$$

$$P_k = \sup_{t \in [0,T]} \left((\frac{|v|}{H_1} + \beta M_1) \, |\varphi_0| \ \right).$$

The Lemma 2 is proved.

<u>Proof of Lemma 1.</u> We recall that $\Lambda = \mathscr{B}^{-1} \, W \, \mathscr{B}$ is the diagonal matrix of eigenvalues of W : $\lambda_1 = 0 < \lambda_2 < \ldots < \lambda_N$, and that $b_{i,j}$ [respectively $b^{-1}_{i,j}$] are the coefficients of the matrix $\mathscr{B}$ [resp. $\mathscr{B}^{-1}$]. We have introduced

$$\psi^M = \mathscr{B}^{-1} \, \psi, \, \xi^M = \mathscr{B}^{-1}\xi, \, C^M = \mathscr{B}^{-1} \, C, \text{ etc}$$

and from (C.7) and (C.8), we deduce that $\psi^M(.,t) \in H^1_M$ where

$$H^1_M = H^1_0 \times \left\{ \psi'^M_k + C^M_k, \, \psi'^M_k \in H^1_0(\Omega), \, \int (\psi'^M_k + C^M_k) \, d\omega = 0 \right\}^{N-1}$$

From (C.1), we obtain for $k \neq 1$

$$\text{(C.9}_k) \qquad \frac{\partial}{\partial t} \left(\Delta\psi'^M_k - \lambda_k \, \psi'^M_k - \lambda_k \, C^M_k\right) = \sum_j b^{-1}_{k,j} \, G_j \, .$$

Let us introduce $\varphi^0 = (\varphi_1^0, \ldots, \varphi_N^0)$ such that

(C.10) $$\begin{aligned} &\Delta\varphi^0 - \Lambda\, \varphi^0 = 1 && \text{in } \Omega \\ &\varphi^0 = 0 && \text{on } \partial\Omega. \end{aligned}$$

Observe that (C.10) gives us

(C.11) $$\int \Delta\varphi_k^0 - \lambda_k \int \varphi_k^0 = \text{mes }(\Omega).$$

It easy to prove (Bernier [3]) that

(C.12) $$\left| \int \Delta\varphi_k^0 \right| > 0\,.$$

Now, we multiply (C.9$_k$) by φ_k^0 and integrate over Ω. We obtain

$$\int \frac{\partial}{\partial t}(\Delta\psi_k^M - \lambda_k\, \psi'^M_k - \lambda_k\, C_k^M)\,.\,\varphi_k^0\, d\omega = \sum_{j=1}^{N} b^{-1}_{k,j} \int G_j\,.\varphi_k^0\, d\omega\,.$$

We rewrite the left-hand side E on the following form :

$$E = \frac{\partial}{\partial t} \int (\Delta\psi_k^M - \lambda_k\, \psi'^M_k - \lambda_k\, C_k^M)\,.\,\varphi_k^0\, d\omega\,.$$

$$E = \frac{\partial}{\partial t} \int \psi'^M_k.(\Delta\varphi_k^0 - \lambda_k\, \varphi_k^0) - \lambda_k\, C_k^M.\varphi_k^0\, d\omega\,.$$

$$E = \frac{\partial}{\partial t}\left(\int \psi'^M_k.1\, d\omega - \lambda_k\, C_k^M \int \varphi_k^0\, d\omega \right) \qquad \text{thanks to (C.10).}$$

$$E = \frac{\partial}{\partial t}\left(-\,\text{mes }(\Omega)\, C_k^M - \lambda_k\, C_k^M \int \varphi_k^0\, d\omega \right) \qquad \text{since } \psi^M \in H^1_M.$$

$$E = \frac{\partial}{\partial t}\left(-\, C_k^M \int \Delta\varphi_k^0\, d\omega \right) \qquad \text{thanks to (C.11).}$$

and then

$$-\frac{\partial}{\partial t}\left(C_k^M \int \Delta\varphi_k^0\, d\omega \right) = \sum_j b^{-1}_{k,j} \int G_j.\varphi_k^0\, d\omega\,.$$

We now use the result of the Lemma 2 to obtain

$$\left|\frac{\partial C_k^M}{\partial t}\right| \left| \int \Delta\varphi_k^0 \right| \le \sum_j |b^{-1}_{k,j}| \left(q_j\, |\Delta\psi_j(t)| + P_j \right).$$

(C.12), $C_1^M = 0$ and the equality $C = \mathcal{B}\, C^M$ allow us to conclude.

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